Creative microscoping

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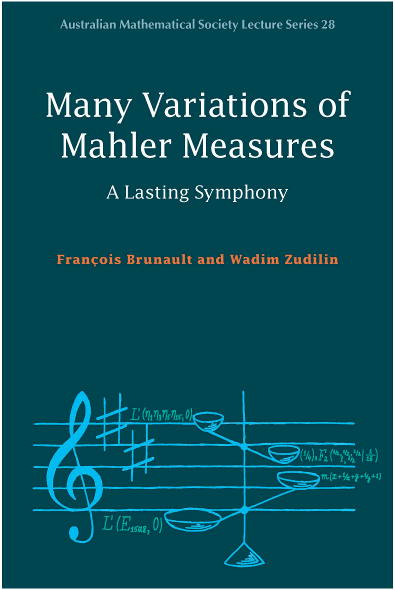
Mahler measures

The (multi-variable) Mahler measure of a polynomial was introduced by Kurt Mahler in the 1960s to give a simple proof of some inequalities for heights in Gelfond's method.

Since then it found remarkable and profound connections with almost any corner of mathematics — see the book.



Ramanujan's mathematics

In 1914, Ramanujan produced a list of rapidly convergent series to $1/\pi$; an example is

$$\sum_{k=0}^{\infty}(8k+1)\frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}}=\frac{2\sqrt{3}}{\pi}.$$

Though Ramanujan's collection serves as an historical background, the principal target of my talk will be the 'impractical' convergence formula

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)A_k=\frac{2}{\pi}, \quad \text{where } A_k=\frac{\binom{2k}{k}^3}{2^{6k}},$$

established by Bauer long before Ramanujan, in 1859.

Bauer's series & Van Hamme's congruences

One natural way of seeing the sequence A_k of the coefficients of the series is via Pochhammer's symbol (also known as shifted factorial)

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \prod_{j=0}^{k-1}(a+j) \quad \text{for } k = 0, 1, 2, \dots,$$

so that Bauer's identity can be stated as

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} = \frac{2}{\pi}.$$

In 1996, Van Hamme noticed a p-adic counterpart of the sum in the form of the (so-called) supercongruences

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \equiv p(-1)^{(p-1)/2} \pmod{p^3} \quad \text{for primes } p > 2,$$

proved in 2008 by Mortenson (and reproved by many since then).

Hypergeometric sums and supercongruences

Though all those proofs of the evaluation

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} = \frac{2}{\pi}$$

and (super)congruences

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \equiv p(-1)^{(p-1)/2} \pmod{p^3} \quad \text{for primes } p > 2$$

share certain similarities, they do not display intrinsic reasons for the two to be related.

Dealing with numerous examples of such duality, we have found with Victor Guo such reasons and used them to prove the above observation and its many other re-incarnations.

Creative q-microscoping

We are for

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac12)_k^3}{k!^3}=\frac{2}{\pi},$$

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac12)_k^3}{k!^3}\equiv p(-1)^{(p-1)/2}\pmod{p^3}\quad\text{for primes } p>2.$$

The key idea is that such identities and congruences follow from a different analysis of the *same* q-hypergeometric identity. This q-identity has more parameters (and their choice correspond to the 'creative' part of the method) and the congruences come from the asymptotics at *all* roots of unity (the 'microscoping' part).

In what follows I will explain the ingredients, in particular, highlight what those q-analogues are and how useful they can be.

q-notation

In order to state those q-analogues we need to familiarise ourselves with standard q-hypergeometric notation. We deal with q inside the unit disc, $|q| < 1$, and define $(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)$ and the related q-Pochhammer symbol by

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty} = \prod_{j=0}^{k-1}(1 - aq^j) \quad \text{for } k = 0, 1, 2, \dots,$$

for non-negative integers n, so that

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n \quad \text{and} \quad \lim_{q\to 1} \frac{(q;q)_\infty (1-q)^{1-a}}{(q^a;q)_\infty} = \Gamma(a).$$

Furthermore, we introduce the q-numbers and q-binomial coefficients as

$$[n] = [n]_q = \frac{1-q^n}{1-q} \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}},$$

so that they correspond to n and $\binom{n}{m}$, respectively, in the limit as $q \to 1$.

q-Bauer sum & q-Van Hamme congruences

Theorem 1. The following identity is true:

$$\sum_{k=0}^{\infty}[4k+1]_q\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}(-1)^kq^k=\frac{(q^2;q^4)_\infty^2(-q^3;q^4)_\infty^2}{(1-q)\,(-q;q^4)_\infty^2(q^4;q^4)_\infty^2}.$$

Theorem 2. Let n be a positive odd integer. Then

$$\sum_{k=0}^{(n-1)/2}[4k+1]_q\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}(-1)^kq^k\equiv\frac{[n]_{q^2}(-q^3;q^4)_{(n-1)/2}}{(-q^5;q^4)_{(n-1)/2}}(-q)^{(1-n)/2}$$

$$\begin{cases}\pmod{\Phi_n(q)\Phi_n(q^2)^2} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q^2)^3} & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Here and in what follows, $\Phi_n(q)$ denotes the n-th cyclotomic polynomial,

$$\Phi_n(q)=\prod_{\substack{j=1\\ \gcd(j,n)=1}}^{n}(q-e^{2\pi ij/n})\in\mathbb{Z}[q].$$

Meta-q-congruences

Theorem 3. Let $n > 1$ be an odd integer. Then

$$\sum_{k=0}^{(n-1)/2} \frac{(1-q^{4k+1})\,(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k}{(1-q)\,(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}\,(-1)^k q^k$$
$$\equiv \frac{(1-q^{2n})\,(-q^3;q^4)_{(n-1)/2}}{(1-q^2)\,(-q^5;q^4)_{(n-1)/2}}\,(-q)^{-(n-1)/2}$$
$$\begin{cases} \pmod{\Phi_n(q)(1-aq^{2n})(a-q^{2n})} & \text{if } n \equiv 1 \pmod 4, \\ \pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

This theorem implies Theorem 2. Indeed, the denominator of the left-hand side related to a is the factor $(aq^4;q^4)_{n-1}(q^4/a;q^4)_{n-1}$; its limit as $a \to 1$ is relatively prime to $\Phi_n(q^2)$, since n is odd. On the other hand, the limit of $(1-aq^{2n})(a-q^{2n})$ as $a \to 1$ has the factor $\Phi_n(q^2)^2$. Thus, letting $a \to 1$ we see that the congruence is true modulo $\Phi_n(q)\Phi_n(q^2)^2$ (or $\Phi_n(q^2)^3$, respectively).

q-Dixon sum

The following summation formula is a (special case of an) old classics known as q-Dixon sum.

Theorem 4. We have

$$\sum_{k=0}^{\infty} \frac{(1-q^{4k+1})\,(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k}{(1-q)\,(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}\,(-1)^k q^k$$
$$= \frac{(q^2;q^4)_\infty(q^6;q^4)_\infty(-aq^3;q^4)_\infty(-q^3/a;q^4)_\infty}{(-q;q^4)_\infty(-q^5;q^4)_\infty(aq^4;q^4)_\infty(q^4/a;q^4)_\infty}.$$

Taking $a=1$, one gets immediately Theorem 1.

Taking $a=q^{2n}$ (or $a=q^{-2n}$), one gets a sum that terminates on the left from $k=n>(n-1)/2$ and (after manipulations) the product

$$\frac{(1-q^{2n})\,(-q^3;q^4)_{(n-1)/2}}{(1-q^2)\,(-q^5;q^4)_{(n-1)/2}}\,(-q)^{-(n-1)/2}$$

on the right. This implies the congruences modulo $a-q^{2n}$ and $1-aq^{2n}$.

q-Microscope I

The remaining — microscoping — part of Theorem 3 is verifying

$$\sum_{k=0}^{(n-1)/2} \frac{(1-q^{4k+1})\,(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k}{(1-q)\,(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}\,(-1)^k q^k \equiv 0$$

modulo $\Phi_n(q)$ or $\Phi_n(q^2)$ depending on whether $n \equiv 1$ or $3 \pmod 4$.

This is equivalent to verifying that the sum $\sum_{k=0}^{(n-1)/2} c_q(k)$ vanishes at any n-th (resp., $2n$-th) root of unity $q=\zeta$, where

$$c_q(k) = \frac{(1-q^{4k+1})\,(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k}{(1-q)\,(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}\,(-1)^k q^k.$$

Notice that the product on the right in Theorem 4,

$$\frac{(q^2;q^4)_\infty(q^6;q^4)_\infty(-aq^3;q^4)_\infty(-q^3/a;q^4)_\infty}{(-q;q^4)_\infty(-q^5;q^4)_\infty(aq^4;q^4)_\infty(q^4/a;q^4)_\infty},$$

is absolutely bounded as $q \to \zeta$.

q-Microscope II

For $n \equiv 1 \pmod 4$ we can use the symmetry $c_\zeta(k) = -c_\zeta(\frac{n-1}{2} - k)$ to conclude on the desired $\sum_{k=0}^{(n-1)/2} c_\zeta(k) = 0$.

In the case $n \equiv 3 \pmod 4$, write the equality in Theorem 4 as

$$\sum_{\ell=0}^{\infty} c_q(\ell n) \sum_{k=0}^{n-1} \frac{c_q(\ell n + k)}{c_q(\ell n)} = \frac{(q^2;q^4)_\infty (q^6;q^4)_\infty (-aq^3;q^4)_\infty (-q^3/a;q^4)_\infty}{(-q;q^4)_\infty (-q^5;q^4)_\infty (aq^4;q^4)_\infty (q^4/a;q^4)_\infty}.$$

Consider the limit as $q \to \zeta = -e^{2\pi i j/n}$ with $(j,n)=1$ radially, that is, $q = r\zeta$ where $r \to 1^-$. On the left-hand side we get

$$\lim_{q\to\zeta} \frac{c_q(\ell n + k)}{c_q(\ell n)} = \frac{c_\zeta(\ell n + k)}{c_\zeta(\ell n)} = c_\zeta(k) \quad \text{and} \quad \lim_{q\to\zeta} c_q(\ell n) = \frac{(\frac12)_\ell}{\ell!}.$$

The divergence of $\sum_{\ell=0}^{\infty} \frac{(\frac12)_\ell}{\ell!}$ and boundedness of the right (product) side then implies that $2\sum_{k=0}^{(n-1)/2} c_\zeta(k) = \sum_{k=0}^{n-1} c_\zeta(k) = 0$.

How creative microscoping works

Here is the summary of how the method works:

$$
\begin{array}{ccccc}
\text{Theorem 4} & \underset{a=1}{\Longrightarrow} & \text{Theorem 1} & \underset{q\to1}{\Longrightarrow} & \text{Bauer's formula} \\
{\scriptstyle a=q^{\pm2n}\ \&\ q\to\zeta}\Big\Downarrow & & & & \\
\text{Theorem 3} & \underset{a\to1}{\Longrightarrow} & \text{Theorem 2} & \underset{q\to1}{\Longrightarrow} & \text{Van Hamme's congruences}
\end{array}
$$

The top of this scheme — Theorem 4 — comes essentially for free from the Gasper–Rahman book nicknamed the q-Bible among the specialists in combinatorics and hypergeometric functions.
Many further entries from the book lead to remarkable (and quite difficult!) congruences, so that the q-Bible turns out to be a treasury book for number theory as well.

A q-bonus: one can consider the limit as $q \to -1$ in Theorems 1 and 2.

Theorems 1 and 2 revisited

Recall the theorems with q replaced with $-q$.

Theorem 1. The following identity is true:

$$\sum_{k=0}^{\infty} \frac{(1+q^{4k+1})\,(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3}\, q^k = \frac{(q^2;q^4)_\infty^2 (q^3;q^4)_\infty^2}{(1+q)\,(q;q^4)_\infty^2 (q^4;q^4)_\infty^2}.$$

Theorem 2. Let n be a positive odd integer. Then

$$\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})\,(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3}\, q^k \equiv \frac{[n]_{q^2}(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}}\, q^{(1-n)/2}$$

$$\begin{cases} \pmod{\Phi_n(-q)\Phi_n(q^2)^2} & \text{if } n \equiv 1 \pmod 4, \\ \pmod{\Phi_n(q^2)^3} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Taking the limit as $q \to 1$ in the theorems we arrive at the following results.

Another limit

The following are true:

$$\sum_{k=0}^{\infty} \frac{(\frac12)_k^3}{k!^3} = \frac{\Gamma(1/4)^4}{4\pi^3} \quad \text{and}$$

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac12)_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p(1/4)^4 \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ -\dfrac{p^2}{16}\,\Gamma_p(1/4)^4 \pmod{p^3} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

The sum is an innocent hypergeometric summation from the literature, while the congruences (even modulo p^3 for $p \equiv 1 \pmod 4$) were shown by Long and Ramakrishna.

The above results can be also cast in the (weaker) form

$$\sum_{k=0}^{\infty} \frac{(\frac12)_k^3}{k!^3} = \frac{8L(f,1)}{\pi} \quad \text{and} \quad \sum_{k=0}^{(p-1)/2} \frac{(\frac12)_k^3}{k!^3} \equiv a(p) \pmod{p^2},$$

where $a(p)$ is the p-th Fourier coefficient of $f = q\prod_{m=1}^{\infty}(1-q^{4m})^6$.

Other applications

The method of creative telescoping demonstrates its potentials for a variety of many other congruences, making their proofs mere exercises.
Many existing (tricky and cumbersome!) proofs of supercongruences have become obsolete.
Examples include p-adic counterparts of Ramanujan's series for $1/\pi$, like

$$\sum_{k=0}^{\infty}(8k+1)\frac{(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k}{k!^3\,9^k}=\frac{2\sqrt{3}}{\pi}$$

mentioned in the beginning, and some partial versions for (now famous!) Guillera's formulas for $1/\pi^2$.
One of the latest achievements is a general framework for (q-analogues of) Dwork-type supercongruences, with many instances experimentally recorded by Swisher.
It is already clear that the method is capable of proving more than we can even expect at this moment.

Dwork-type supercongruences

Examples from our last work with Guo include the supercongruences

$$\sum_{k=0}^{(p^r-1)/2}(8k+1)\frac{(\frac14)_k(\frac12)_k(\frac34)_k}{k!^3\,9^k}\equiv p\left(\frac{-3}{p}\right)\sum_{k=0}^{(p^{r-1}-1)/2}(8k+1)\frac{(\frac14)_k(\frac12)_k(\frac34)_k}{k!^3\,9^k}$$

modulo p^{3r} for primes $p>3$ (which underlie Ramanujan's formula from the previous slide) and $r=1,2,\dots,$ as well as

$$\sum_{k=0}^{(p^r-1)/2}(-1)^k(4k+1)\frac{(\frac12)_k^3}{k!^3}\equiv p\left(\frac{-1}{p}\right)\sum_{k=0}^{(p^{r-1}-1)/2}(-1)^k(4k+1)\frac{(\frac12)_k^3}{k!^3}\pmod{p^{3r}}$$

for primes $p>2$ (which generalize our Theorem 2). We further have

$$\sum_{k=0}^{(p^r-1)/2}\frac{(\frac12)_k^2}{k!^2}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{(p^{r-1}-1)/2}\frac{(\frac12)_k^2}{k!^2}\pmod{p^{2r}}$$

for primes $p>2$, the case $r=1$ conjectured by Rodriguez Villegas and proved by Mortenson, and also in my joint work with Chan and Long.

All supercongruences in a uniform way?

It does not look, at least at this moment, that all hypergeometric (super)congruences can be done via the creative microscope.
For example, in joint work with Long, Tu and Yui we prove the supercongruences

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{4})_k(\frac{1}{3})_k(\frac{2}{3})_k(\frac{3}{4})_k}{k!^4} \equiv b(p) \pmod{p^3} \quad \text{for } p>3,$$

where $\sum_{n=1}^{\infty} b(n)q^n = q\prod_{m=1}^{\infty}(1-q^{3m})^8$, and 13 other alike in a uniform way using completely different techniques.
This lacks a microscopic approach, however the q-deformation behind

$$\frac{S_{r+1}}{S_r} \equiv \frac{S_r}{S_{r-1}} \pmod{p^{3r}} \quad \text{for } r=1,2,\dots, \quad \text{where}$$

where $S_r = \displaystyle\sum_{k=0}^{(p^r-1)/2} \frac{(\frac{1}{4})_k(\frac{1}{3})_k(\frac{2}{3})_k(\frac{3}{4})_k}{k!^4}$, is experimentally detected.

Summary

Here is again the summary of how the creative microscoping works:

$$\begin{array}{ccccc}
\text{Theorem 4} & \underset{a=1}{\Longrightarrow} & \text{Theorem 1} & \underset{q\to1}{\Longrightarrow} & \text{Bauer's formula} \\
{\scriptstyle a=q^{\pm2n}\ \&\ q\to\zeta}\Downarrow & & & & \\
\text{Theorem 3} & \underset{a\to1}{\Longrightarrow} & \text{Theorem 2} & \underset{q\to1}{\Longrightarrow} & \text{Van Hamme's congruences}
\end{array}$$

Questions?

Be (q-)hypergeometric!