Modular Forms and Invariant Theory

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Modular Forms

Everybody should know modular forms:

$$f : \mathfrak{H} \to \mathbf{C} \qquad \text{holomorphic such that}$$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 = \mathrm{SL}(2,\mathbf{Z})$, so that

$$f = \sum_{n \geq 0} a(n) q^n \quad \text{with } q = e^{2\pi i \tau}.$$

Here k is the weight of f.

$M_k(\Gamma_1)$ the vector space of such f.

Examples: Eisenstein series

$$E_4 = 1 + 240 \sum \sigma_3(n)\, q^n$$

$$E_6 = 1 - 504 \sum \sigma_5(n)\, q^n$$

We have

$$R_1 = \oplus_k M_k(\Gamma_1) = \mathbf{C}[E_4, E_6]\,.$$

For the algebraic geometer: sections of powers of the Hodge bundle. If $\pi : \mathcal{X}_1 \to \mathcal{A}_1$ is the universal elliptic curve then

$$\mathbf{E} = \pi_* \Omega^1_{\mathcal{X}_1/\mathcal{A}_1}$$

and it extends over the compactification $\tilde{\mathcal{A}}_1$. Over $\mathbf{C}$

$$\mathbf{E} = \Gamma_1 \backslash \mathfrak{H} \times \mathbf{C}$$

under the action $(\tau, z) \mapsto (\gamma(\tau), (c\tau + d)z)$. If $E = \mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$ then dz changes under γ to $dz/(c\tau + d)$.

We have

$$M_k(\Gamma_1) = H^0(\tilde{\mathcal{A}}_1, \mathbf{E}^{\otimes k})\,.$$

We can generalize this. We have the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension $g > 1$; it carries a universal p.p. abelian variety $\pi : \mathcal{X}_g \to \mathcal{A}_g$.

Over $\mathbf{C}$ we have

$$\mathcal{A}_g(\mathbf{C}) = \Gamma_g \backslash \mathfrak{H}_g$$

where $\Gamma_g = \mathrm{Sp}(2g, \mathbf{Z}) = \mathrm{Aut}(\mathbf{Z}^{2g}, \langle\, , \,\rangle)$, with $(\mathbf{Z}^{2g}, \langle\, , \,\rangle)$ the lattice with basis

$e_1, \ldots, e_g, f_1, \ldots, f_g$ with

$$\langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle, \quad \langle e_i, f_j \rangle = \delta_{ij}$$

and

$$\mathfrak{H}_g = \{\tau \in \mathrm{Mat}(g \times g, \mathbf{C}) : \tau^t = \tau, \mathrm{Im}(\tau) > 0\}$$

An element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g$ acts on $\mathfrak{H}_g$ by

$$\tau \mapsto (a\tau + b)(c\tau + d)^{-1}.$$

We then can look at holomorphic functions

$$f : \mathfrak{H}_g \to \mathbf{C}, \quad \text{with}$$

$$f(\gamma(\tau)) = \det(c\tau + d)^k f(\tau) \qquad (\gamma \in \Gamma_g)$$

Again we have a graded ring

$$R_g = \oplus_k M_k(\Gamma_g)$$

the ring of Siegel modular forms of degree g.

In the 1960s Igusa determined the structure of R_2:

$$R_2 = \mathbf{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}]/(\chi_{35}^2 - \cdots)$$

For $g = 3$ Tsuyumine gave 34 generators in 1986 and recently Lercier-Ritzenthaler reduced that to 19.

The Hodge bundle $\mathbf{E} = \mathbf{E}^{(g)} = \pi_* \Omega^1_{\mathcal{X}_g/\mathcal{A}_g}$ has rank g. Over $\mathbf{C}$

$$\mathbf{E} = \Gamma_g \backslash \mathfrak{H}_g \times \mathbf{C}^g$$

under the action $(\tau, z) \mapsto (\gamma(\tau), (c\tau + d)z)$.

Thus we can do more: if given an irrep

$$\rho : \mathrm{GL}(g, \mathbf{C}) \to \mathrm{GL}(W)$$

then we have the bundle $\mathbf{E}_\rho$ obtained by applying ρ to the transition functions of $\mathbf{E}$. Over $\mathbf{C}$ its sections are given by holomorphic maps

$$f : \mathfrak{H}_g \to W$$

satisfying $f(\gamma(\tau)) = \rho(c\tau + d)f(\tau)$.

How to describe these modular forms?

A modular form F of genus g has a Fourier expansion

$$F = \sum_{n \geq 0} a(n)\, q^n \quad \text{with} \quad q^n := e^{2\pi i \mathrm{Tr}(n\tau)}$$

with $a(n) \in W$ and n running over symmetric $g \times g$ matrices with $n \geq 0$, $2n$ integral and with even diagonal.

In our work we used the Torelli map to construct Siegel modular forms

$$\mathcal{M}_g \xrightarrow{t} \mathcal{A}_g, \quad C \mapsto \mathrm{Jac}(C)\,.$$

Use that $\mathcal{M}_g$ is close to $\mathcal{A}_g$ for $g = 2$ and $g = 3$:

$$\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$$

and

$$\mathcal{M}_3 \xrightarrow{2:1} \mathcal{A}_3$$

The Case $g = 2$

Pull back $\mathbf{E}$ to $\mathcal{M}_2$; it extends to $\overline{\mathcal{M}}_2$.

A curve of genus 2 is hyperelliptic: $y^2 = f(x)$, with $\deg(f) = 6$, $\mathrm{discr}(f) \neq 0$ (in $\mathrm{char} \neq 2$). We write

$$f = \sum_{i=0}^{6} a_i \, x_1^{6-i} x_2^i$$

that is, $f \in \mathrm{Sym}^6(V)$ with $V = \langle x_1, x_2 \rangle$. The group $\mathrm{GL}(V) = \mathrm{GL}(2)$ acts and $\mathcal{M}_2$ is a stack quotient

$$[\mathcal{Y}^0 / \mathrm{GL}(V)]$$

with $\mathcal{Y}^0 \subset \mathcal{Y} = \mathrm{Sym}^6(V) \otimes \det(V)^{-2}$, with $\mathcal{Y}^0$ referring to $\mathrm{disc}(f) \neq 0$.

We twisted by $\det(V)^{-2}$ so that $-\mathrm{id}_V$ acts by $(x, y) \mapsto (x, -y)$.

A curve $y^2 = f$ comes with differentials

$$dx/y,\ xdx/y$$

and $\mathrm{GL}(V)$ acts by the standard representation.

We get

$$[\mathcal{Y}^0/\mathrm{GL}(V)] \cong \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$$

The pull back of $\mathbf{E}_2$ to $\mathcal{Y}^0$ is the equivariant bundle V.

And pull back of $\det(\mathbf{E})$ to $\mathrm{Sym}^6(V)$ is $\det(V)^3$ because we twisted.

Invariant Theory

An invariant for the action of $\mathrm{GL}(V) = \mathrm{GL}(2, \mathbf{C})$ on $\mathrm{Sym}^6(V)$ is a polynomial in the coefficients $a_0, \ldots, a_6$ of f invariant under $\mathrm{SL}(2, \mathbf{C})$. Example: the discriminant $\mathrm{discr}(f)$.

Invariants form a ring (Bolza, Clebsch,..)

$$I = \mathbf{C}[A, B, C, D, E]/(E^2 = ...)$$

Now $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ gives us a map

$$R_2 \longrightarrow I \qquad \text{with}$$

$\psi_4 \mapsto B$, $\psi_6 \mapsto C - AB$, $\chi_{10} \mapsto D$, $\chi_{12} \mapsto AD$.

Igusa (1962) made such a map using theta functions and Thomae's formulas.

Not every invariant gives a modular form; e.g. $A \longmapsto \chi_{12}/\chi_{10}$. Indeed $\mathcal{M}_2 \subset \mathcal{A}_2$; complement is $\mathcal{A}_{1,1}$, zero locus of χ_{10}. We get

$$R_2 \longrightarrow I \longrightarrow R_2[1/\chi_{10}]$$

But we also have covariants, that is, the invariants for the action of $\mathrm{SL}(2, \mathbf{C})$ on

$$V \oplus \mathrm{Sym}^6(V)$$

Alternatively, if $U \hookrightarrow \mathrm{Sym}^d(\mathrm{Sym}^6(V))$ is an equivariant embedding or, equivalently, if we have a map of $\mathrm{GL}(V)$-representations

$$\mathbf{C} \xrightarrow{\phi} \mathrm{Sym}^d(\mathrm{Sym}^6(V)) \otimes U^\vee$$

then $\Phi = \phi(1)$ is a covariant. If U has highest weight (λ_1, λ_2) then Φ is a form of degree d in $a_0, \ldots, a_6$ and degree $\lambda_1 - \lambda_2$ in x_1, x_2.

Example: $d = 1$: $\mathrm{Sym}^6(V) \cong \mathrm{Sym}^6(V)$. Then $\Phi = f = \sum a_i\, x_1^{6-i} x_2^i$, our "universal sextic".

Example: $d = 2$. In this case we can write $\mathrm{Sym}^2(\mathrm{Sym}^6(V))$ as

$$U[12, 0] + U[10, 2] + U[8, 4] + U[6, 6]$$

with corresponding four covariants

$$f^2, \mathrm{Hessian}, \ldots,$$

$$-240\, a_0 a_6 + 40\, a_1 a_5 - 16\, a_2 a_4 + 6\, a_3^2$$

Covariants form a ring $\mathcal{C}$, with 26 generators (Cayley, Grace and Young, 19th century)

An irrep of $\mathrm{GL}(2,\mathbf{C})$ is $\mathrm{Sym}^j(\mathrm{St})\otimes\det(\mathrm{St})^{\otimes k}$ with St the standard representation.

For $g=2$ the R_2-module

$$M=\oplus_{j,k}M_{j,k}(\Gamma_2)$$

can be made into a ring.

Proposition 1. *Pullback via $\mathcal{Y}^0\to\mathcal{M}_2\hookrightarrow\mathcal{A}_2$ defines maps*

$$M\xrightarrow{\mu}\mathcal{C}\xrightarrow{\nu}M[1/\chi_{10}]$$

with $\nu\circ\mu=\mathrm{id}_M$.

Apply this to the universal sextic $f\in\mathcal{C}$. What is $\nu(f)$?

$$f\mapsto \text{merom. section of}\quad \mathrm{Sym}^6(\mathbf{E})\otimes\det(\mathbf{E})^{-2}$$

This form can be identified: we have six odd thetas $\vartheta_\epsilon(\tau, z)$, hence six gradients

$$(\partial\vartheta_\epsilon/\partial z_1, \partial\vartheta_\epsilon/\partial z_2)\,(\tau, 0)$$

The product of these is a modular form $\chi_{6,3}$ with character on Γ_2. We then have

$$\nu(f) = \chi_{6,-2} = \chi_{6,3}/\chi_5 \qquad \text{with } \chi_5^2 = \chi_{10}$$

Write $\chi_{6,-2}$ with dummy variables X_1, X_2 as

$$\chi_{6,-2} = \sum_{i=0}^{6} \alpha_i\, X_1^{6-i} X_2^i$$

Here α_i is meromorphic on $\mathfrak{H}_2$.

Recall that a covariant of degree d is a form in $a_0, \ldots, a_6, x_1, x_2$ and of degree d in the a_i.

Theorem 1. *The map*

$$\nu : \mathcal{C} \to M \otimes R_2[1/\chi_{10}]$$

is given by the substitution $a_i \mapsto \alpha_i$, $x_i \mapsto X_i$ in a covariant.

This is an extremely effective method.

The website smf.compositio.nl gives Fourier series for all cases where $\dim S_{j,k}(\Gamma_2) = 1$.

We determined the module structure for $\oplus_k M_{j,k}(\Gamma_2, \epsilon)$ for $j = 2, 4, 6, 8$ and 10. (Here ϵ is the unique quadratic character of Γ_2.)

We formulated a conjecture about the vanishing of $S_{j,2}(\Gamma_2)$ and gave evidence for it ($S_{j,2}(\Gamma_2) = (0)$ for $j \leq 52$).

The method works also in positive characteristic.

Theorem 2. *In char* 3 *the even weight subring* $R_2^{\mathrm{ev}}(\mathbf{F}_3)$ *is generated by forms of weights* $2, 10, 12, 14, 36$ *and has the form*

$$R_2^{\mathrm{ev}}(\mathbf{F}_3) = \mathbf{F}_3[\psi_2, \chi_{10}, \psi_{12}, \chi_{14}, \chi_{36}]/J$$

with J *generated by*

$$\psi_2^3\chi_{36} - \chi_{10}^3\psi_{12} - \psi_2^2\chi_{10}\chi_{14}^2 + \chi_{14}^3 .$$

Moreover

$$R_2(\mathbf{F}_3) = R_2^{\mathrm{ev}}(\mathbf{F}_3)[\chi_{35}]/(\chi_{35}^2 - \cdots) .$$

For the binary sextic

$$f = a_0 x^6 + a_1 x^5 + \cdots + a_6$$

the invariant

$$A = -240 a_0 a_6 + 40 a_1 a_5 - 16 a_2 a_4 + 6 a_3^2$$

becomes

$$A = a_1 a_5 - a_2 a_4 \pmod 3$$

and regular. It gives rise to the Hasse invariant (of weight $p - 1 = 2$).

Theorem 3. *The ring $R_2(\mathbf{F}_2)$ is generated by modular forms of weights $1, 10, 12, 13, 48$ with one relation of weight 52:*

$$R_2(\mathbf{F}_2) = \mathbf{F}_2[\psi_1, \chi_{10}, \psi_{12}, \chi_{13}, \chi_{48}]/(R)$$

with

$$R = \chi_{13}^4 + \psi_1^3 \chi_{10} \chi_{13}^3 + \psi_1^4 \chi_{48} + \chi_{10}^4 \psi_{12} \,.$$

The Case $g = 3$

Torelli $t : \mathcal{M}_3 \longrightarrow \mathcal{A}_3$ is of degree 2, ramified along the hyperelliptic locus $\mathcal{H}_3$. The modular form $\chi_{18} = \prod_\epsilon \vartheta[\epsilon](\tau, 0)$ has divisor

$$\mathcal{H}_3 + 2D \qquad \text{in } \widetilde{\mathcal{A}}_3$$

with $D = \tilde{\mathcal{A}}_3 - \mathcal{A}_3$.

We pull back the Hodge bundle $\mathbf{E}$ and similarly the $\mathbf{E}_\rho$ and extend to $\overline{\mathcal{M}}_3$. We define

$$T_\rho = H^0(\overline{\mathcal{M}}_3, \mathbf{E}_\rho),$$

the space of Teichmüller modular forms of weight ρ. (Here ρ is an irrep of $\mathrm{GL}(3, \mathbf{C})$.)

Example: $\chi_9 = \sqrt{\chi_{18}} \in T_9$ (with $\rho = \det^9$). Due to Ichikawa.

There is an involution ι (coming from $\mathcal{M}_3 \xrightarrow{2:1} \mathcal{A}_3$) and we have eigenspaces

$$T_\rho = T_\rho^+ \oplus T_\rho^- .$$

We can identify

$$T_\rho^+ \cong S_\rho(\Gamma_3),$$

while T_ρ^- is the space of *genuine* Teichmüller forms; we have

$$\chi_9 \, T_\rho^- \subset S_{\rho'}(\Gamma_3) \quad \text{with } \rho' = \rho \otimes \det^9$$

We have another description of the non-hyperelliptic part of $\mathcal{M}_3$:

$$\mathcal{M}_3^{nh} \cong [\mathcal{Y}^0/\mathrm{GL}(V)]$$

$$\mathcal{Y}^0 \subset \mathrm{Sym}^4(V) \otimes \det{}^{-1}(V).$$

with $V = \langle x, y, z \rangle$. We look at the space of smooth ternary quartics $\mathcal{Y}^0$ inside all ternary quartics $\mathcal{Y}$. (We twisted such that $c \cdot \mathrm{id}_V$ acts as $c \cdot \mathrm{id}_{\mathcal{Y}}$.)

For the invariant theory (of $\mathrm{GL}(V)$) we have to look at concomitants. Take an equivariant map of $\mathrm{GL}(V)$-reps

$$U \hookrightarrow \mathrm{Sym}^d(\mathrm{Sym}^4(V))$$

equivalently

$$\varphi : \mathbf{C} \longrightarrow \mathrm{Sym}^d(\mathrm{Sym}^4(V)) \otimes U^\vee$$

Then $\Phi = \varphi(1)$ is a concomitant. The concomitants form a module $\mathcal{C}$ over the ring I of invariants for $\mathrm{GL}(3, \mathbf{C})$.

For the modular forms we have a module

$$\Sigma = \oplus_\rho T_\rho$$

of vector-valued Teichmüller modular forms over the ring T of scalar-valued Teichmüller modular forms.

The pull back of $\mathbf{E}$ under $\mathcal{Y}^0 \to \mathcal{M}_3^{nh}$ is $\mathcal{Y}^0 \times V$. This gives us

$$\begin{array}{ccccc} T & \longrightarrow & I & \longrightarrow & T[1/\chi_9] \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma & \longrightarrow & \mathcal{C} & \xrightarrow{\nu} & \Sigma[1/\chi_9] \end{array}$$

Modular forms give concomitants, concomitants give meromorphic modular forms.

Where does the concomitant $f \in \mathcal{C}$, the universal quartic, go under ν?

$$f \longmapsto \chi_{4,0,-1},$$

a meromorphic section of

$$\mathrm{Sym}^4(E) \otimes \det(E)^{-1}$$

on $\overline{\mathcal{M}}_3$ with

$$\chi_{4,0,-1} \cdot \chi_9 = \chi_{4,0,8} \in S_{4,0,8}(\Gamma_3),$$

a holomorphic Siegel modular form.

How to get it? Note $\dim S_{4,0,8}(\Gamma_3) = 1$.

We take the Schottky form for $g = 4$ (of weight 8 vanishing on the Torelli locus) and develop it along

$$\mathfrak{H}_3 \times \mathfrak{H}_1 \hookrightarrow \mathfrak{H}_4$$

The first non-zero term in the Taylor expansion is

$$\chi_{4,0,8} \otimes \Delta \in S_{4,0,8}(\Gamma_3) \otimes S_{12}(\Gamma_1)$$

Its Fourier expansion starts as follows: write $q_i = e^{2\pi i \tau_{ii}}$ $(i = 1, 2, 3)$ and
$u = e^{2\pi i \tau_{12}}$, $v = e^{2\pi i \tau_{13}}$, $w = e^{2\pi i \tau_{23}}$)

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ (v-1)^2(w-1)^2/vw \\ (u-1)(v-1)(w-1)(-1+1/vw+1/uw-1/uv) \\ (u-1)^2(w-1)^2/uw \\ 0 \\ (u-1)(v-1)(w-1)(-1+1/vw-1/uw+1/uv) \\ (u-1)(v-1)(w-1)(-1-1/vw+1/uw+1/uv) \\ 0 \\ 0 \\ 0 \\ (u-1)^2(v-1)^2/uv \\ 0 \\ 0 \end{pmatrix} q_1 q_2 q_3 + \cdots ,$$

We describe the map $\nu : \mathcal{C} \to \Sigma[1/\chi_9]$.

Write the universal quartic f as $\sum a_I x^I$. Then write $\chi_{4,0,8}$ as

$$\chi_{4,0,8} = \sum_I \alpha_I X^I$$

as a ternary quartic with dummy variables X_1, X_2, X_3. The α_I are holomorphic on $\mathfrak{H}_3$, given by their Fourier expansion.

A concomitant is a polynomial in the a_I, x_1, x_2, x_3 and u_1, u_2, u_3 with $u_1 = x_2 \wedge x_3$, $u_2 = x_1 \wedge x_3$ and $u_3 = x_2 \wedge x_3$.

Theorem 4. *The map ν is given by substituting α_I/χ_9 for a_I.*

Is the result holomorphic?

Theorem 5. *Let c be a concomitant of degree d (say d odd). Let $v(c)$ be its order of vanishing along the locus of double conics. Then $\nu(c)\chi_9$ is a Siegel modular form vanishing with order $v(c) - (d-1)/2$ along the hyperelliptic locus.*

Example: Let c be the discriminant, an invariant of degree $d = 27$. Now

$$\nu(c)\chi_9 = \chi_{18}$$

vanishes with order 1 along $\mathcal{H}_3$, the hyperelliptic locus. So c vanishes with order 14 along the locus of double conics (confirms a result of Aluffi-Cuckierman).

In principle we can describe all modular forms of genus 3.

Example. For $d = 2$ we find the decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^4(V))$ as

$$V[8,0,0] + V[6,2,0] + V[4,4,0]\,.$$

The component $V[4,4,0]$ defines a concomitant. It is a form of degree 2 in the a_I and degree 4 in u_1, u_2, u_3, the generators of $\wedge^2 V$. It gives a meromorphic Siegel modular form of weight $(0,4,-2)$. It becomes holomorphic after multiplication by χ_9^2, hence a form in $S_{0,4,16}$ vanishing twice at ∞.

Example. The catalecticant is an invariant of degree $d = 6$ associated to

$$V[8,8,8] \subset \mathrm{Sym}^6(\mathrm{Sym}^4(V)).$$

It defines a Siegel modular form of weight 56 vanishing with order 6 at ∞ and order 16 along $\mathcal{A}_{2,1}$. It confirms a result of Ottaviani-Sernesi who view the catalecticant as a section $\mathcal{O}(56\lambda - 6\delta_0 - 16\delta_1)$ on $\overline{\mathcal{M}}_3$.

website: smf.compositio.nl

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