Calculus 3 Final Sample Exam 5

courtesy of Dr. Inna Sysoeva, University of Pittsburgh, adapted

Problem 1. Find the points on the cone $z^2 = x^2 + y^2$ closest to the point $(1, \sqrt{3}, 4)$.

Problem 2. Express the volume of the tetrahedron with vertices at points $(0,0,0),\ (1,0,0),\ (0,2,0),\ (0,0,3)$ as a triple integral. Write out the limits of integration explicitly, but do not evaluate.

Problem 3. Write the equation of any plane tangent to the cone $z^2 = x^2 + y^2$.

Problem 4. Use Stokes' theorem to evaluate $\int\limits_C \mathbf{F} \cdot d\mathbf{r}$, where $F(x,y,z) = e^x\mathbf{i} + (x^2+y)\mathbf{j} + z\mathbf{k}$, and C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first quadrant oriented counterclockwise as viewed from above.

Problem 5. Consider the helix $r(t) =< \cos t, \sin t, t >,\ 0 \le t \le \pi$.

a) Find the length of this helix.

b) If the helix has density z at point (x,y,z), what is its mass?

Problem 6. Let D be the region $D = \{(x,y,z) : x^2 + y^2 + z^2 \le 2,\ z \ge x^2 + y^2\}$ and S be the surface of D with outward pointing normal.

a) Calculate the volume of D.

b) Let $\mathbf{F}(x,y,z) =< yz^2, 3x + z\cos x, x^2y^3 >$. Use the divergence theorem and the result in part a) to find $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$.

Problem 7. Let S be the lateral surface of the cone $z^2 = x^2 + y^2$, between $z = 0$ and $z = 1$ with outward orientation, and let $\mathbf{F}(x,y,z) =< -y, x, 0 >$. Denote by I the value of the integral $\iint\limits_S curl(\mathbf{F}) \cdot d\mathbf{S}$.

a) Compute $curl(\mathbf{F})$ and $div(\mathbf{F})$.

b) Find I by direct evaluation.

c) Find I by using Stokes' theorem.

d) Find I by use of the divergence theorem. (Hint: Cover the top of the cup by a flat circular disk.)

Problem 8. Find two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$, such that $\mathbf{v} \cdot \mathbf{w} = 0$ and both $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to the line $x = y = z$.

Problem 9. Find the limit, if it exists:

$$\lim_{(x,y)\to(0,0)} \frac{2009x^2 - y^2}{1 - \sqrt{1 + 2009x^2 - y^2}}$$

Problem 10. Let $\mathbf{F}(x,y,z) =< xy - 3, -y^2, yz - z >$. Evaluate in the simplest way possible the surface integral $\int\limits_S \int \mathbf{F} \cdot \mathbf{n} dS$, where S is the boundary of the unit cube $\{(x,y,z) : 0 \le x, y, z \le 1\}$, and $\mathbf{n}$ denotes the inward unit normal to S.