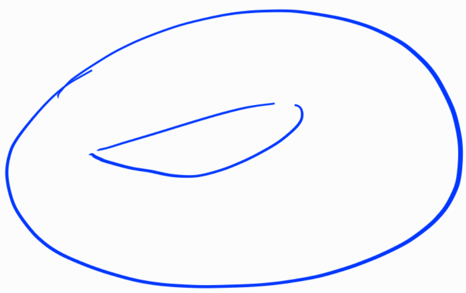
Toric Varieties

Torus — real torus / algebraic torus



$T^2 = T \times T$

$T = \mathbb{R}/\mathbb{Z} \cong S^1$

unit circle

Algebraic torus:

$$(\mathbb{C}^*)^n \supset T^n$$

$$\begin{array}{ccc} (\mathbb{C}^*)^n & & T^n \\ \parallel & & \parallel \\ \{(x_1 \cdots x_n)\} & & \forall |x_i| = 1 \end{array}$$

$$(\mathbb{C}^*)^n = T^n \oplus \mathbb{R}^n$$

$$z \quad \left(\frac{z}{|z|}, \ln|z|\right)$$

$(\mathbb{C}^*)^n$ is a <u>reductive group</u>

<u>Functions</u> on $(\mathbb{C}^*)^n$

$$\mathbb{C}^* \qquad \cancel{\frac{2x+\pi}{x-3}}$$

$$\shortparallel$$

$$(x)$$

Laurent polynomials

$$a_k x^k + \dots + a_m x^m$$

$$k \le m$$

$$2x^{-3} + 3x^5$$

$(\mathbb{C}^*)^n$ $\qquad \sum_{i_1, \dots, i_n} a_i x_1^{i_1} \dots x_n^{i_n}$

$x_1 \dots x_n$

N

A representation of G is $f : G \to GL_N(\mathbb{C})$

If $G = (\mathbb{C}^*)^n$, we only consider f

such that all entries are Laurent polynomials.

Definition. An algebraic group is reductive if every complex algebraic representation of it is semi-simple:

if we have a subrepresentation, then there is an invariant complement.

Note. $\mathbb{C}$ is not a reductive group.

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad x \in \mathbb{C}$$

$$\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + x_2 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \subsetneq \mathbb{C}^2$$

no invariant complement

Ex. $f : \mathbb{C}^* \to GL_2(\mathbb{C})$

$$f(z) = \begin{pmatrix} 1 & \ln|z| \\ 0 & 1 \end{pmatrix}$$

One idea

If $f: G \to GL_n(\mathbb{C})$

and $\exists$ p.d. Hermitian

form H, s.t. $\forall g \in G$

$(f(g)\vec{u}, f(g)\vec{v})_H = (\vec{u}, \vec{v})_H$

Then if $V \subset \mathbb{C}^n$ is invariant,

so is $V^\perp = \{\vec{v} \in \mathbb{C}^n \mid \forall \vec{u} \in V \ (\vec{u}, \vec{v})_H = 0\}$

$\vec{v} \in$

$(u, f(g)\vec{v}) = (f(g)\vec{w}, f(g)\vec{v}) = (\vec{w}, \vec{v})$

$= 0$

$u = f(g)\underbrace{(f(g^{-1})u)}_{\vec{w} \in V}$

Weyl's trick

$(\mathbb{C}^*)^n \supsetneq T^n \cong (T^1)^n$

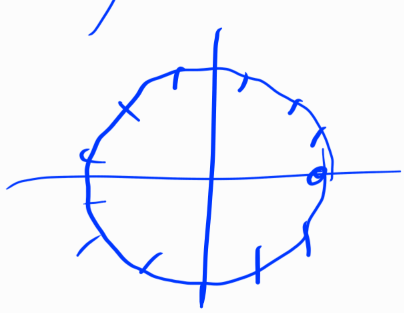
$\underline{\text{compact}}$

T^n is Zariski dense in $(\mathbb{C}^*)^n$

Classification of representations of $(\mathbb{C}^*)^n$

Inside $\mathbb{C}^*$ we have many roots of unity: $\forall k$

$\exists\, \mu_k \subset \mathbb{C}^*$ $\forall z \in \mu_k$ $z^k = 1$



In $(\mathbb{C}^*)^n$ we have

$\forall k$ $\quad (x_1, \dots, x_n) = g$

s.t. $\forall i$ $x_i^k = 1$

<u>Claim.</u> For each g as above for every $f : (\mathbb{C}^*)^n \to GL_n(\mathbb{C})$

$f(g)$ is diagonalizable, and eigenvalues are roots of 1

$$\begin{array}{l} f(g^k) = (f(g))^k \\ \parallel \\ \underline{I} \end{array}$$

$f(g)$ for all g, all k

<u>Thm:</u> Any collection of commuting diagonalizable matrices is simultaneously diagonalizable.

$$\begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_1 & & & & & \\ & & & \lambda_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_2 & & \\ & & & & & & \ddots & \\ 0 & & & & & & & \lambda_n \end{pmatrix}$$

$$f : (\mathbb{C}^*)^n \to GL_n(\mathbb{C})$$

$\exists$ a basis in $\mathbb{C}^n$

$\forall\, \beta$ s.t. $\beta^k = (1, 1, \dots, 1)$

$f(\beta)$ is diagonal

Thm. If a Laurent polynomial is 0 on all elements of the form $\beta^{k}=1$ for some k, then this polynomial is identically zero.

Proof.

1) If $h(x) \in \mathbb{C}[x]$ s.t. for ∞ x $h(x)=0$

Then $h=0$.

2) If $h(x)$ is a Laurent polynomial — " —

$\exists m$ $x^{m} \cdot h(x)$ is in $\mathbb{C}[x]$

. . . .

3) $h(x_1, \dots, x_n)$ a Laurent polynomial, $\forall b$ with $b^k = 1$ $h(b) = 0$.

$\forall x_2, \dots x_n$ that are roots of unity $h(x, x_2, \dots x_n) = 0$ because for ∞ x_1 it is 0.

So all coefficients of it are 0 ... induction

<u>Thm.</u> Every $f : \overline{(\mathbb{C}^*)^n \to \mathbb{C}^*}$ is of the form $f((x_1 \dots x_n)) = x_1^{k_1} \cdot \dots \cdot x_n^{k_n}$

where $\forall i$ $k_i \in \mathbb{Z}$

<u>Proof.</u> Enough to show this for $n=1$

$(\mathbb{C}^*)^n$ is generated by

$$(1 \ldots \underbrace{*}_{x_i} 1 \ldots 1) \xmapsto{f} X_i^{k_i}$$

$$\mathbb{C}^* \to \underline{\mathbb{C}^*}$$

Laurent polynomial

$$X \mapsto C \cdot X^k$$

$$1 \mapsto 1 \Rightarrow C = 1$$