

$$\sigma \in \sum_{\uparrow} \text{fan}$$

σ is a cone in N
 closed convex rational
 polyhedral, strictly convex

$$X_{\sigma} = \text{mspec}(\hat{\sigma} \cap M)$$

A face of σ : τ

$$\sigma \cap \{f=0\}$$

$$\text{where } \underline{f \in \hat{\sigma}}$$

We can choose $f \in \hat{\sigma} \cap M$

Lemma.

$C[\hat{\sigma}_f \cap M]$ is
a localization of

$C[\hat{\sigma} \cap M]$ by

the $\{1, (f), (f)^2, \dots\}$

$$(f) = x_1^{a_1} \dots x_n^{a_n}$$

$$f = \sum_{i=1}^n a_i e_i \quad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in M \quad \begin{matrix} x^f = (f) \\ \parallel_n \end{matrix}$$

Proof. We will

show:

$$\forall g \in \hat{\sigma}_f \cap M$$

$$\exists k \in \mathbb{N} \text{ s.t. } \frac{g}{k} = h - kf$$

where $h \in \hat{\sigma} \cap M$

σ is a rational polyhedral cone, generated by

$$\vec{v}_1, \dots, \vec{v}_n \in N$$

$$g(\vec{v}_i) \in \mathbb{Z}$$

$$\text{If } \forall i \quad g(\vec{v}_i) \geq 0,$$

then $g \in \sigma$, take it any

f is 0 on the face,

but is positive on

all generators of σ

that are not on the

face.

So if we add to

g a large enough

multiple of f ,

then $g(\vec{v}_i) + Kf(\vec{v}_i) \geq 0$

for all $\vec{v}_i$ not in $\hat{\sigma}_f$

Since $g \in \hat{\sigma}_f$,

$$g(\vec{v}_i) \geq 0 \quad \forall \vec{v}_i \in \hat{\sigma}_f$$

So for large enough K

$$h = g + Kf \in \hat{\sigma}$$

$$\underline{x}^g = \frac{x^h}{(\underline{x}^f)^K} \quad \text{in}$$

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

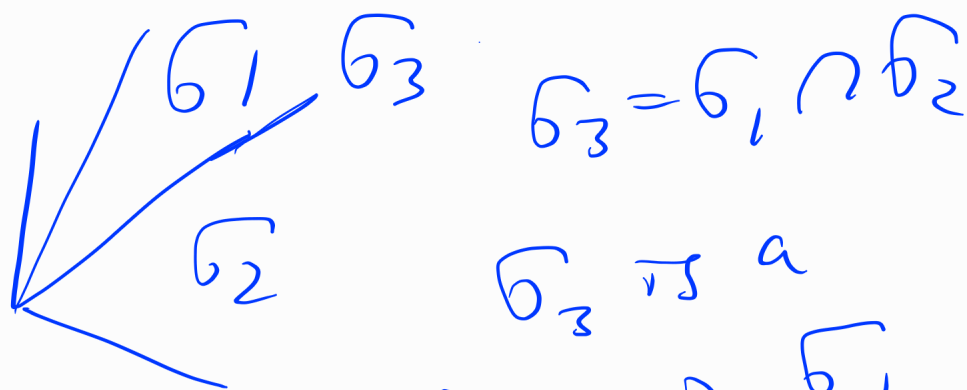
$$\underline{\mathbb{C}[\hat{\sigma} \cap M]} \subseteq \underline{\mathbb{C}[\hat{\sigma}_f \cap M]}$$

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$R \xrightarrow{\subseteq} R_f$$

localization
by $\{f, f^2, \dots\}$

$$\begin{array}{ccc} m \text{ Spec}(R) & \longleftrightarrow & m \text{ Spec}(R_f) \\ \left(\begin{array}{l} m \subset R \\ \text{maximal} \end{array} \right) & & \left(\begin{array}{l} m \subset R \\ \text{maximal} \\ f \notin m \end{array} \right) \end{array}$$



σ_3 is a
face in σ_1
and in σ_2

$$\begin{array}{ccc} X_{\sigma_1} & & X_{\sigma_2} \\ & \searrow & \swarrow \\ & X_{\sigma_3} & \end{array}$$

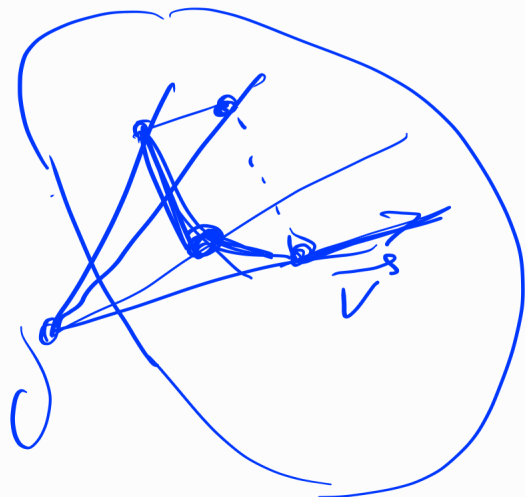
We give X_{σ_1} and X_{σ_2}
by X_{σ_3}

Useful observation

Suppose $\sigma \cap N$ is a
strictly convex polyhedral

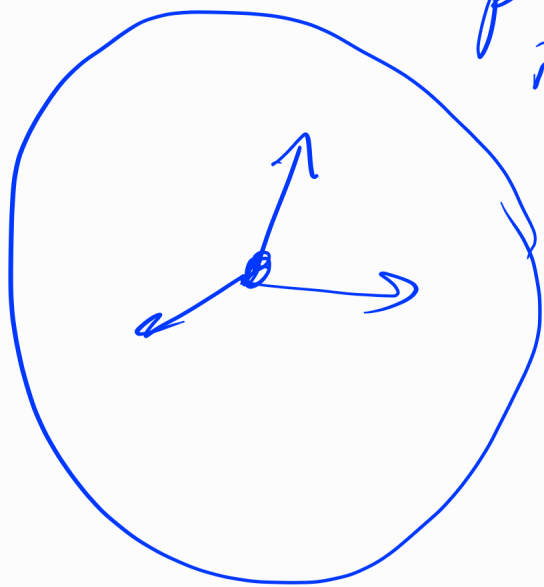
cone.

Def. $\vec{v} \in \sigma$ is a ray
of σ if $\vec{v} \neq \vec{0}$
if $\vec{v} = \vec{v}_1 + \vec{v}_2$, $\vec{v}_i \in \sigma$
then $\vec{v}_1 = c_1 \vec{v}$ $\vec{v}_2 = c_2 \vec{v}$.



- Thm. 1) Every strictly convex $\mathcal{C}$ is generated by its rays
- 2) Every set of generators of $\mathcal{C}$ must contain a non-zero multiple of every ray.

Ex. 1)



2)



Proof. 2) Suppose $\vec{v}$ is

a ray.

$$\vec{b} = \langle \vec{v}_1, \dots, \vec{v}_k \rangle$$

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_k \vec{v}_k$$

$$\forall c_i \geq 0.$$

$$\vec{v} = c_1 \vec{v}_1 + \underbrace{\left(\dots \right)}_{\vec{b}}$$

$c_1 \vec{v}_1$ is a multiple of $\vec{v}$

Either $c_1 \neq 0$, we are done.

or $c_1 = 0$, then we look

for $c_2 \vec{v}_2 \dots$

1) Suppose $\underbrace{\langle \vec{v}_1, \dots, \vec{v}_m \rangle}_{\text{all rays}}, \vec{v}_{m+1}, \dots, \vec{v}_k \rangle = \vec{b}$

$\vec{V}_k$ is not a ray.

$$\begin{aligned}\vec{V}_k &= \sum_{i=1}^{k-1} a_i \vec{V}_i + \sum b_i \vec{V}_i \\ &= \sum_{i=1}^{k-1} (a_i + b_i) \vec{V}_i + (a_k + b_k) \vec{V}_k\end{aligned}$$

If $a_k + b_k < 1$, then

$$\vec{V}_k (1 - (a_k + b_k)) = \sum_{i=1}^{k-1} (a_i + b_i) \vec{V}_i$$

$\vec{V}_k$ can be removed

If $a_k + b_k > 1$

$$\vec{V}_k (1 - (a_k + b_k)) = \sum_{i=1}^{k-1} (a_i + b_i) \vec{V}_i$$

$-\vec{V}_k \in \mathcal{C}$ So $\mathcal{C}$ is not strictly convex

If $a_k + b_k = 1$

$$\vec{0} = \sum_{i=1}^{k-1} a_i \vec{V}_i + \sum_{i=1}^{k-1} b_i \vec{V}_i$$

$\xrightarrow{\frac{1}{\vec{0}}} \vec{V}_0$ $\xrightarrow{-\vec{V}_0}$

$\vec{V}_0 \in \mathcal{C}, -\vec{V}_0 \in \mathcal{C}$

This contradicts $\bar{b}$ being strictly convex $\square$

Faces of strictly convex polyhedra are

A ray of a face must be a ray of the original cone.

$\bar{b}$ face $\bar{b}_f$

$$\bar{b} = \bigcup_{f \geq 0} \bar{b}_f$$

$f \geq 0$ on $\bar{b}$

$\vec{v} \neq \vec{0}$ a ray of $\bar{b}_f$

$\vec{v} \in \bar{b}_f$ so $\vec{v} \in \bar{b}$

suppose $\vec{v} = \vec{v}_1 + \vec{v}_2, \vec{v}_i \in \bar{b}$

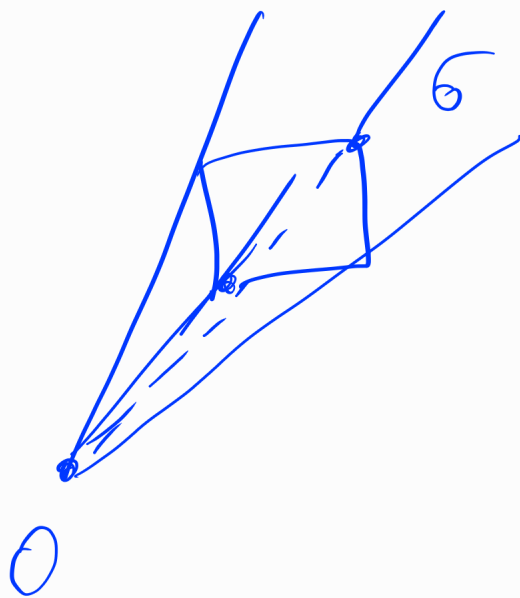
$$f(\vec{v}) = \underbrace{f(\vec{v}_1) + f(\vec{v}_2)}_{\vec{0}}$$

$$f(\vec{v}) \parallel 0 \quad \text{So } f(\vec{v}_1) = f(\vec{v}_2) = 0$$

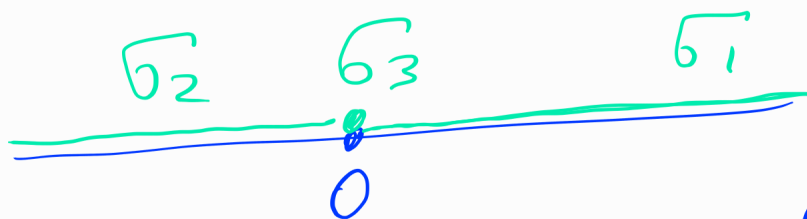
$$\vec{v}_1, \vec{v}_2 \in \mathcal{B}_f$$

$$\text{So } \vec{v}_1 = c_1 \vec{v}, \vec{v}_2 = c_2 \vec{v}$$

because $\vec{v}$ is a ray
in $\mathcal{B}$.



$$(n=1) \quad M=\mathcal{Q}, N=\mathcal{Q} \subseteq \mathbb{R}$$



Possible
 $\{b_3\}$
 $\mathcal{C}^*$

faus:
 $\{b_1, b_3\}$
 $\mathcal{C}$

$\mathcal{C}$
 $\{b_2, b_3\}$
 $\{b_1, b_2, b_3\}$
 $\mathcal{C} \cup \mathcal{P}_1$

