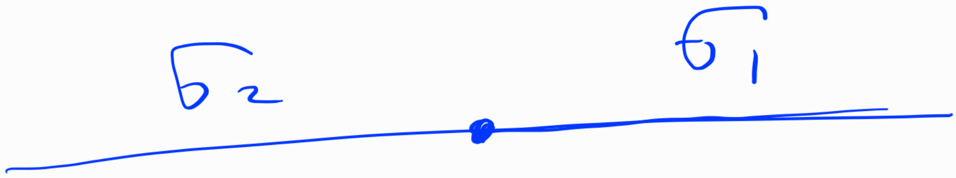
$d = 1$



$N = \mathbb{Z}$

$\sigma_1 = \{ n \mid n \geq 0 \}$

$\sigma_2 = \{ n \mid n \leq 0 \}$

$\sigma_3 = \{ 0 \}$

1. $\Sigma = \{ \sigma_3 \}$
2. $\Sigma = \{ \sigma_3, \sigma_1 \}$
3. $\Sigma = \{ \sigma_3, \sigma_2 \}$
4. $\Sigma = \{ \sigma_3, \sigma_1, \sigma_2 \}$

① $\mathrm{mSpec}(\mathbb{C}[\mathbb{Z}]) = \mathbb{C}^*$

x, x^{-1}

② $\mathrm{mSpec}(\mathbb{C}[\hat{\sigma}_1 \cap M])$

$\sigma_1 = \{n \mid n \geq 0\}$

$\hat{\sigma}_1 = \{m \mid \underline{m \cdot n} \geq 0 \ \forall n \in \sigma_1\}$

$= \{m \mid \underline{m \geq 0}\}$

$M = \mathrm{Hom}(N, \mathbb{Z}) \quad (\cong \mathbb{Z}^d)$

$N = \mathrm{Hom}(M, \mathbb{Z}) \quad (\cong \mathbb{Z}^d)$

$\mathrm{mSpec}(\mathbb{C}[\mathbb{Z}_{\geq 0}]) \cong \mathbb{C}$

x

$\mathbb{C}^* \subset \mathbb{C}$

$\sigma_3 \preccurlyeq \sigma_1$

③ $\sigma_3, \sigma_2, \ldots$

$$\hat{\sigma}_2 = \{ m \in M \mid m \leq 0 \}$$

$$\mathrm{mSpec}\left(\mathbb{C}[x^{-1}]\right) \cong \mathbb{C}$$

$x^{-1} = t$ $\quad$ t variable

④ $\Sigma = \{\sigma_3, \sigma_1, \sigma_2\}$

$$\begin{array}{ccccc} \sigma_1 & \cap & \sigma_2 & = & \sigma_3 \\ | & & | & & | \\ \mathbb{C} & & \mathbb{C} & & \mathbb{C}^* \end{array}$$

σ_1 $\quad$ $t = x^{-1}$ $\quad$ σ_2

$$\mathbb{C} \qquad\qquad \mathbb{C} \cong \mathrm{mSpec}(\mathbb{C}[t])$$

$$\mathbb{C} \cong \mathrm{mSpec}(\mathbb{C}[x]) \supset \mathbb{C}^* \subset \mathbb{C}$$

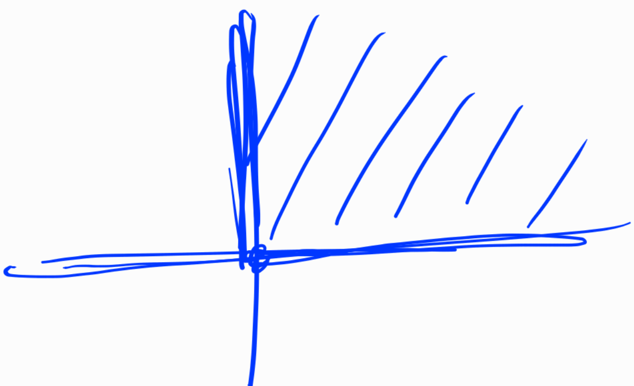
$$\mathbb{C}^* \cong \mathbb{C}_t \setminus \{t = 0\}$$

$$\mathbb{C}^* \cong \mathbb{C} \setminus \{x = 0\}$$

This is $\mathbb{C}P^1$

Ex. $d=2$

$$\sigma = \{(n_1, n_2) \mid n_1 \geq 0,\ n_2 \geq 0\}$$



$$\hat{\sigma} = \{(m_1, m_2) \mid m_1 \geq 0,\ m_2 \geq 0\}$$

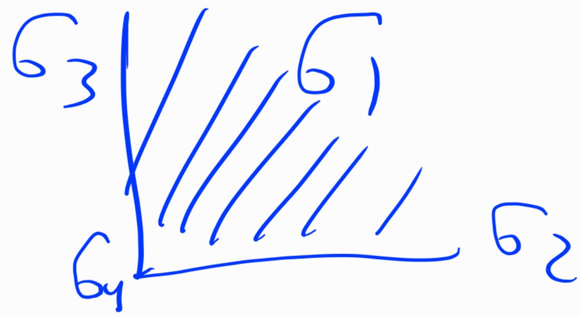
$$(m_1, m_2) \cdot (n_1, n_2) = m_1 n_1 + m_2 n_2$$

$$\begin{cases} (m_1, m_2) \cdot (1, 0) \geq 0 \\ (m_1, m_2) \cdot (0, 1) \geq 0 \end{cases} \qquad \begin{cases} m_1 \geq 0 \\ m_2 \geq 0 \end{cases}$$

$$m\mathrm{Spec}\left(\mathbb{C}[x_1, x_2]\right) \simeq \mathbb{C}^2$$

monomials $\quad x_1^{m_1} x_2^{m_2}, \quad m_1, m_2 \geq 0$

Fan:



$$\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$$

$$\sigma_1 = \{(n_1, n_2) \mid n_1, n_2 \geq 0\}$$

$$\sigma_2 = \{(n_1, 0) \mid n_1 \geq 0\}$$

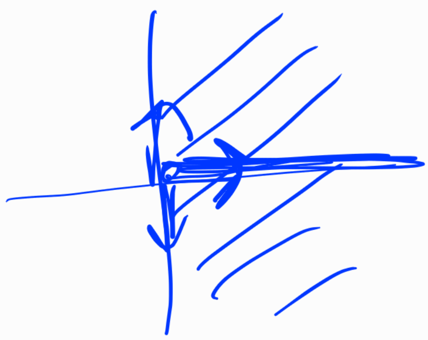
$$\sigma_3 = \{(0, n_2) \mid n_2 \geq 0\}$$

$$\sigma_4 = \{(0,0)\}$$

$$\begin{tikzcd}
 & \overset{\sigma}{\mathbb{C}^2} \ (x_1, x_2) & \\
\overset{\sigma_2}{(x_2 \neq 0)} \ \mathbb{C} \times \mathbb{C}^* \arrow[ur, hook] & & (x_1 \neq 0) \ \mathbb{C}^* \times \mathbb{C} \ \sigma_3 \arrow[ul, hook'] \\
 & \mathbb{C}^* \times \mathbb{C}^* \ \sigma_4 \arrow[ul, hook] \arrow[ur, hook'] & \\
 & (x_1 \neq 0, x_2 \neq 0) &
\end{tikzcd}$$

$\sigma_2 = \{(n_1, 0) \mid n_1 \geq 0\}$

$\hat{\sigma}_2 = \{(m_1, m_2) \mid m_1 \geq 0\}$



$(1,0), (0,1), (0,-1)$

x_1, x_2, x_2^{-1}

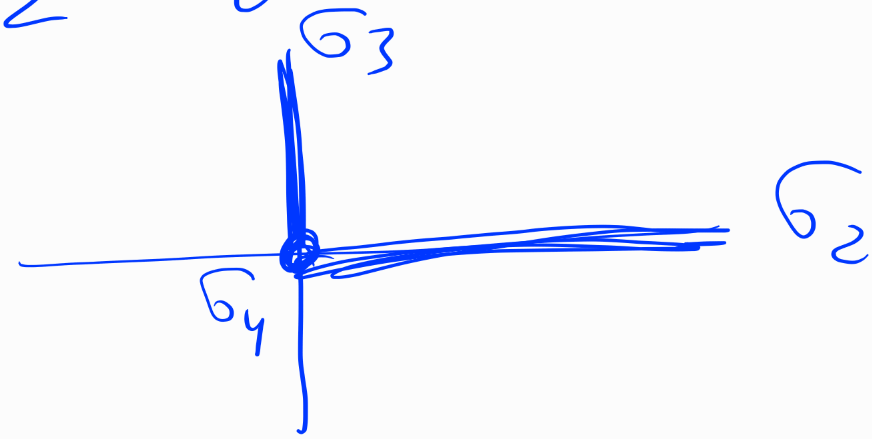
$\mathbb{C}[x_1, x_2, x_2^{-1}]$

$(x_1, x_2) \quad x_2 \neq 0$

This generalizes to any dimension d.

Ex. $(d=2)$

$\Sigma = \{\sigma_2, \sigma_3, \sigma_4\}$

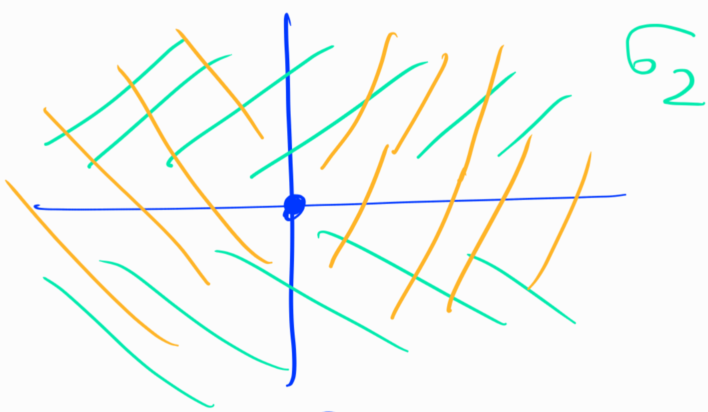


This $\mathbb{C}^2 \setminus \{(0,0)\}$

σ_2 $\mathbb{C} \times \mathbb{C}^*$ $x_2 \neq 0$

σ_3 $\mathbb{C}^* \times \mathbb{C}$ $x_1 \neq 0$

σ_4 $\mathbb{C}^* \times \mathbb{C}^*$ $\begin{cases} x_1 \neq 0, \\ x_2 \neq 0 \end{cases}$



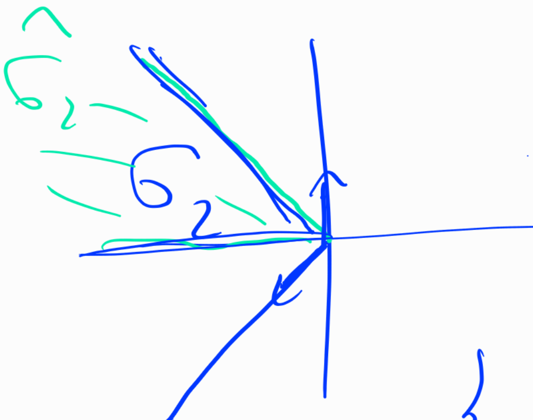
Ex$_1$ $\mathbb{CP}^2$ N

$\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_1 \cap \sigma_2,$
$\sigma_1 \cap \sigma_3, \sigma_2 \cap \sigma_3, \{0\}\}$

$$\hat{\sigma}_1 = \{(m_1, m_2) \mid m_1 \geq 0, m_2 \geq 0\}$$

$\mathbb{C}[x_1, x_2]$ $\qquad$ $\mathbb{C}^2$ $\quad (x_1, x_2)$

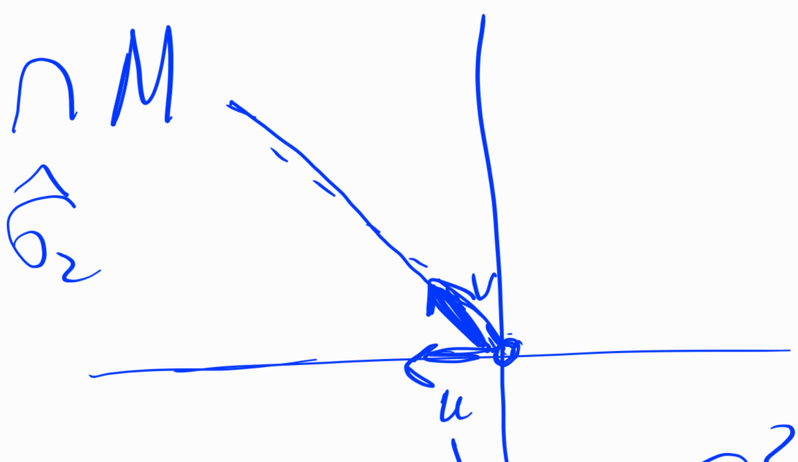
$$\hat{\sigma}_2 = \{(m_1, m_2) \mid m_1 n_1 + m_2 n_2 \geq 0 \ \ \forall (n_1, n_2) \in \sigma_2\}$$



$$\begin{cases} m_1 \cdot 0 + m_2 \cdot 1 \geq 0 \\ m_1(-1) + m_2(-1) \geq 0 \end{cases}$$

$$\begin{cases} m_2 \geq 0 \\ m_1 + m_2 \leq 0 \end{cases}$$

$\hat{\sigma}_2 \cap M$



$m\operatorname{Spec}(\mathbb{C}[u, v])$ $\qquad$ $\mathbb{C}^2$

$$\begin{cases} u = x_1^{-1} \\ v = x_1^{-1} x_2 \end{cases}$$

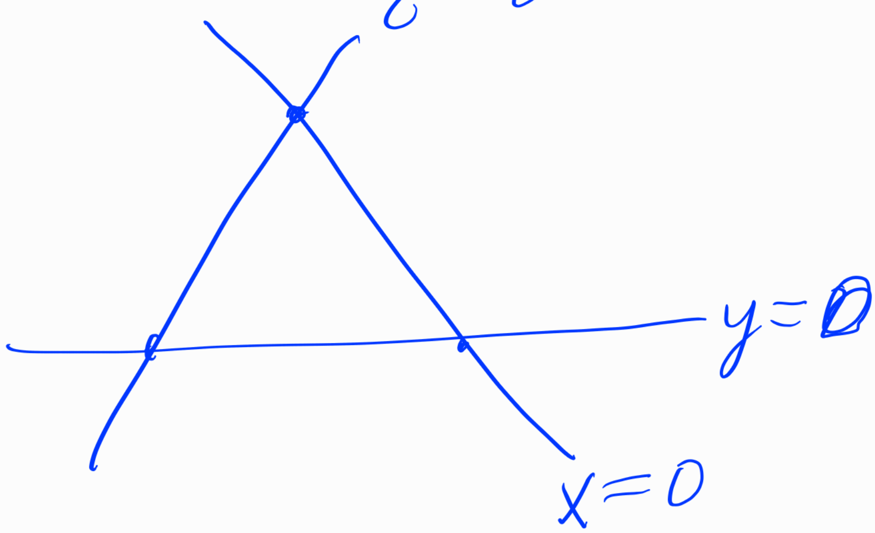
$$(X_1, X_2) \underset{z\neq 0}{\longleftrightarrow} (X_1 : X_2 : 1)$$

$$(x:y:z) \qquad \begin{cases} X_1 = \frac{x}{z} \\ X_2 = \frac{y}{z} \end{cases}$$

$$(x:y:z), \quad x \neq 0$$

$$\longleftrightarrow \left(1 : \underset{\substack{\parallel \\ v}}{\frac{y}{x}} : \underset{\substack{\parallel \\ u}}{\frac{z}{x}}\right)$$

$$\begin{cases} u = X_1^{-1} \\ v = X_1^{-1} X_2 \end{cases}$$

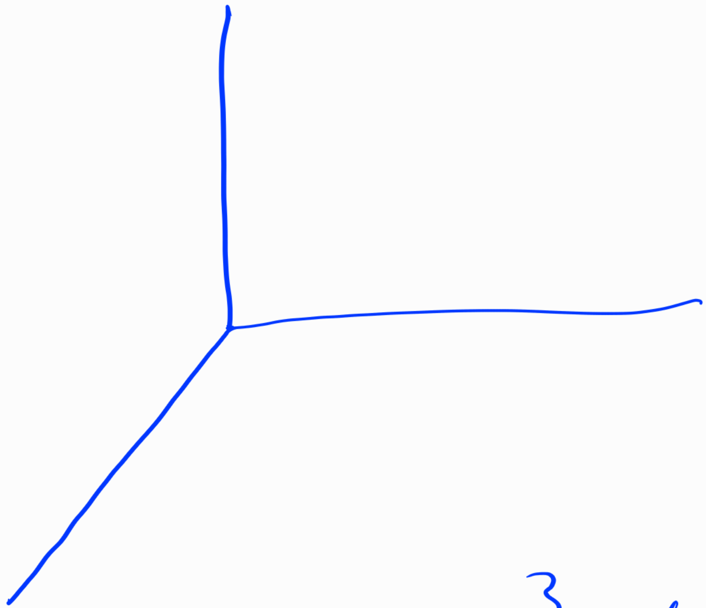


$$(x:y:z)$$

$$\mathbb{CP}^2 \setminus \{x=0\} \simeq \mathbb{C}^2$$

$$\mathbb{CP}^2 \setminus \{y=0\} \simeq \mathbb{C}^2$$

$$\mathbb{CP}^2 \setminus \{z=0\} \simeq \mathbb{C}^2$$



$$(\mathbb{C}^*)^2 \cong (\mathbb{C}^*)^3 / \mathbb{C}^*$$

$$(x_1 : x_2 : 1)$$

$$(\mathbb{C}^*)^2 = \{(x_1, x_2) \mid x_1, x_2 \neq 0\}$$

$(\mathbb{C}^*)^2$ acts on $\mathbb{CP}^2$

$$(x_1, x_2)(x : y : z) = (x_1 x : x_2 y : z)$$