Normal varieties

Def. R (domain) is called integrally closed (= normal) if $\forall f \in Q(R)$, if f is integral over R, then f is in R.

In other words: Suppose $f = \frac{p}{q}$ $(p, q \in R)$ is such that for some $a_{n-1}, a_{n-2}, \dots, a_0 \in R$

$$f^n + a_{n-1} f^{n-1} + \dots + a_1 f + a_0 = 0$$

Then $f \in R$.

Ex. $R = \{a + b\sqrt{3}i \mid a, b \in \mathbb{Z}\}$

$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$2\omega \in R$ $\quad \omega \in Q(R)$

$\omega^2 + \omega + 1 = 0$

Ex. $R = \mathbb{C}[x,y]/(y^2 - x^3 - x^2)$

$f = \frac{\bar{y}}{\bar{x}}$ $\quad \bar{y} \in R$, $\bar{x} \in R$

$\wedge$

$Q(R)$

$y^2 - x^3 - x^2 = 0$

$\Downarrow$

$\left(\frac{y}{x}\right)^2 - x - 1 = 0$

$f^2 - (\bar{x} + 1) = 0$ in $Q(R)$

So f is integral over R

But f is <u>not</u> in R

$$\frac{\bar{y}}{\bar{x}} \overset{\text{in } Q(R)}{=} \bar{z} \iff \bar{y} \overset{\text{in } R}{=} \bar{x}\,\bar{z}$$

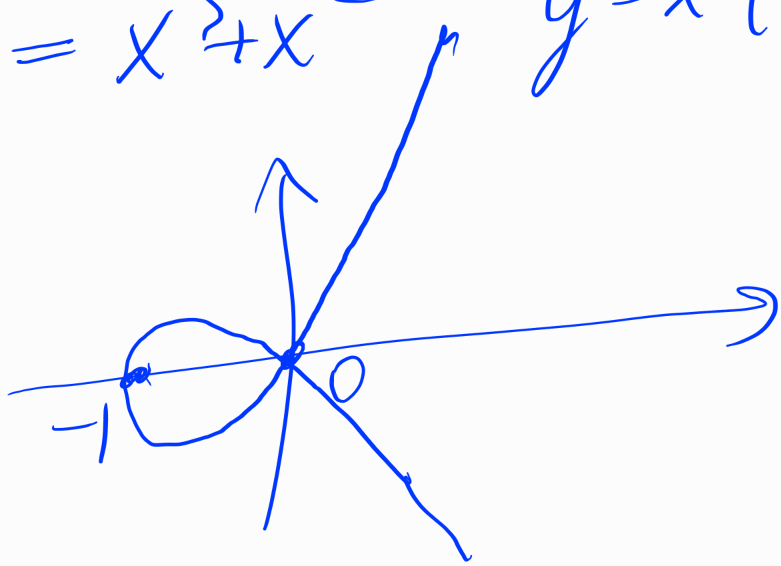
$$\iff y - xz \in (y^2 - x^3 - x^2)$$

$$y - xz = h(x,y)(y^2 - x^3 - x^2)$$

$$y = \underbrace{x \cdot z(x,y)}\ + \underbrace{h(x,y)(y^2 - x^3 - x^2)}$$

Impossible

$$y^2 = x^3 + x^2 \qquad y^2 = x^2(x+1)$$



Need:

$R = \mathbb{C}[\hat{\sigma} \cap M]$ is integrally closed.

Thm $\mathbb{C}[x_1, \dots, x_d]$ is a UFD.

Gauss Lemma.

R is UFD

$p(x), q(x) \in R[x]$

$\begin{cases} \gcd(\text{coeff. of } p) = 1 \\ \gcd(\text{coeff. of } q) = 1 \end{cases}$

Then $\gcd(\text{coeff. of } pq) = 1$

Corollary. If R is UFD then $R[x]$ is also UFD.

Lemma: If R is a UFD, then any localization of R is also UFD.

In particular, $\mathbb{C}[X_1, \ldots, X_d, X_1^{-1}, \ldots, X_d^{-1}]$ is UFD

It is a localization of $\mathbb{C}[X_1 \ldots X_d]$ w.r.t. $(1, X_1 \ldots X_d, (X_1 \ldots X_d)^2, \ldots)$

Valuation

F field.

A valuation on F is $\mathrm{val} : F \to \mathbb{R}_{\geq 0}$ s.t.

val : $f \to v(f)$

1) $v(x) = 0 \iff x = 0$

2) $v(f \cdot g) = v(f) \cdot v(g)$

3) $v(f+g) \le v(f) + v(g)$

Ex. $F = \mathbb{Q}$

$v_1(x) = |x|$

p is a prime.

$$v_p\left(\frac{n}{m}\right) = \begin{cases} 0, & n = 0 \\ \frac{1}{p^k}, & k = ord_p(n) - ord_p(m) \end{cases}$$

Ex. $p = 3$

$$V_3\left(\frac{63}{50}\right) = \frac{1}{9}$$

$ord_3 63 = 2$

$$\begin{cases} v_p(n_1) = \frac{1}{p^{k_1}} \\ v_p(n_2) = \frac{1}{p^{k_2}} \end{cases}$$

$n_1 \equiv 0 \mod p^{k_1}$

$n_2 \equiv 0 \mod p^{k_2}$

Then:

$$v_p(n_1+n_2) \le \max\left(\frac{1}{p^{k_1}}, \frac{1}{p^{k_2}}\right)$$

$$n_1+n_2 \equiv 0 \mod p^{\min(k_1,k_2)}$$

This is an example of a Non-archimedean valuation

<u>Ex.</u> $R = \mathbb{C}[x_1, x_1^{-1}, \dots, x_d, x_d^{-1}]$

$F = Q(R) = \mathbb{C}(x_1 \dots x_d)$

Suppose $(n_1, \dots, n_d) \in N$

<u>Ex.</u> $d=2$ $F = Q(x_1, x_2)$

$(n_1, n_2) = (2, 3)$

For $p(x_1, x_2)$ polynomial

$$v(p) = e^{-\inf_X (2m_1 + 3m_2)}$$

$X = \{$ non-zero monomials $a_{m_1 m_2} x_1^{m_1} x_2^{m_2}$ in $p \}$

For every $\sigma \subset N_{\mathbb{R}}$

we can describe the monomials in

$$R = \mathbb{C}[\hat{\sigma} \cap M]$$

as those monomials cut out by the conditions

$$x_1^{m_1} \cdots \cdots x_d^{m_d} \in R$$

$$\Updownarrow$$

$$v_{r_i}(x_1^{m_1} \cdots \cdots x_d^{m_d}) \leq 1$$

for all valuations that correspond to rays of σ.

$$\hat{\sigma} = \{(m_1 \ldots m_d) \mid \underline{\forall (n_1 \ldots n_d) \in \sigma} \quad m_1 n_1 + \ldots + m_d n_d \geq 0\}$$

$$\hat{\sigma} = \{(m_1 \ldots m_d) \mid \forall i \ (m_1 \ldots m_d) \cdot \vec{r_i} \geq 0\}$$

where $\vec{r_i}$ is a generator of the ray.

Proof of normality
of a toric variety.

$$f \in \mathbb{C}(x_1 \dots x_d)$$

$$\| \quad \frac{p(x_1 \dots x_d)}{q(x_1 \dots x_d)}$$

p, q Laurent polynomials

Suppose f is integral
over $\mathbb{C}[\hat{\sigma} \cap M]$.

① $q = 1$ (upto units)
if q has an irreducible
factor that is not a
Laurent monomial,
we get a contradiction.

$$\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \dots + a_0 = 0$$

(Laurent polys are UFD)

② $f = p$ Laurent poly.

If for some i

$v_{r_i}(p) > 1$ then

$$p^n + (a_{n-1}p^{n-1} + \dots + a_1 p + a_0) = 0$$

is impossible $v_{r_i}|a_i| \leq 1$

$$\mathrm{val}(a_{n-1}p^{n-1} + \dots + a_1 p + a_0)$$

$$\leq (\mathrm{val}\, p)^{n-1}$$

So $\forall r_i$

$$\min_{a_{m_1 \dots m_d} \neq 0 \text{ in } P} (m_1, \dots m_d) \vec{r_i} \geq 0$$

So all monomials in P are in $\mathbb{C}[\hat{\sigma} \cap M]$ $\square$