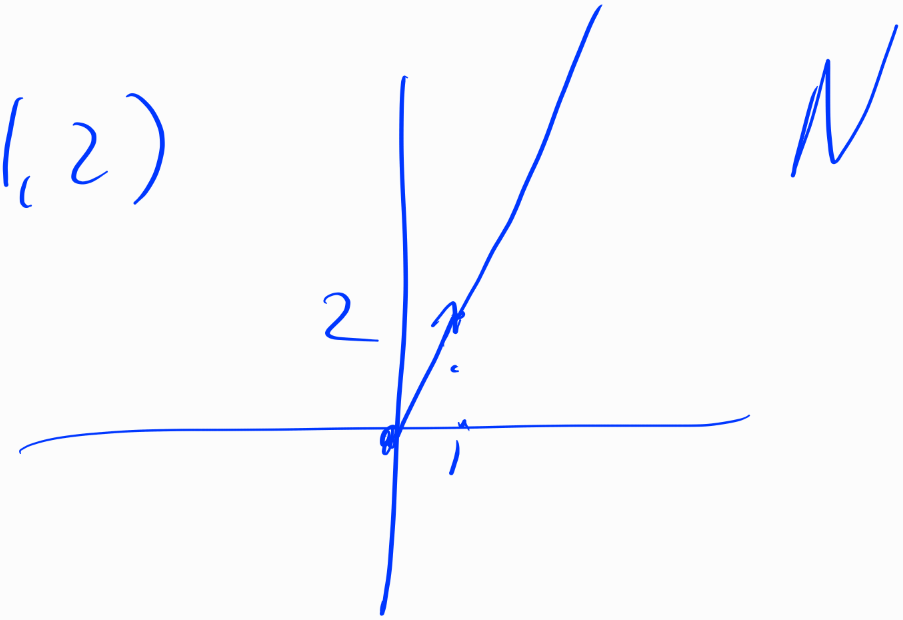
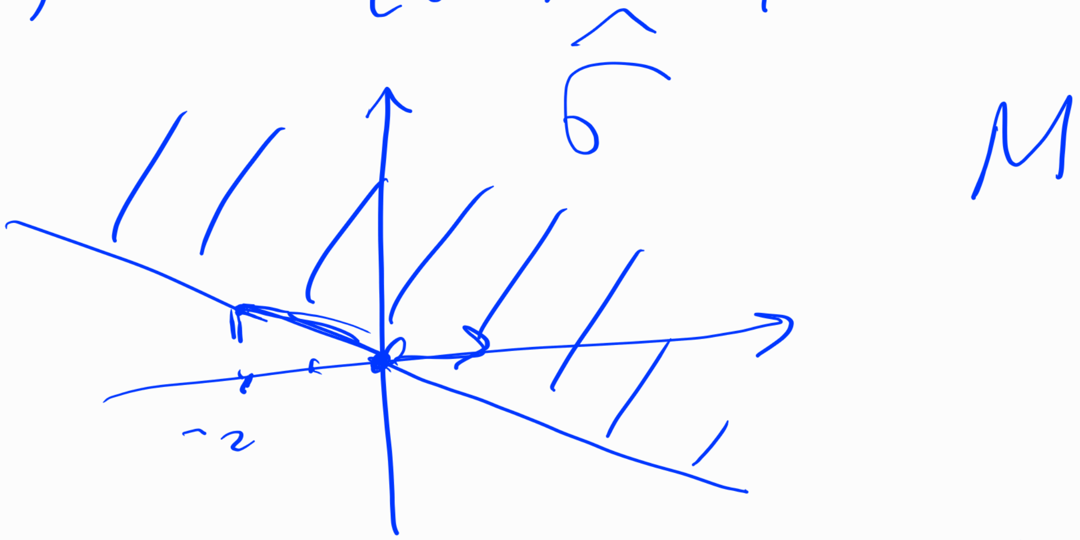
HW

① $(1,2)$



a) $\hat{\sigma} = \{(m_1, m_2) \mid \underline{1\cdot m_1 + 2\cdot m_2 \geq 0}\}$



$\hat{\sigma} = \langle (-2,1), (2,-1), (1,0) \}$

b) $\hat{\sigma} \cap M = \langle \underbrace{\overset{u}{(-2,1)}, \overset{v}{(2,-1)}} \overset{w}{(1,0)} \}$

$\underline{(m_1, m_2) = c\cdot(-2,1) + d(1,0)}$

where $c \in \mathbb{Z}$, $d \geq 0$

$\underline{d = 1 \cdot m_1 + 2 m_2}$

$\underline{(1, 0)} \rightsquigarrow 1 \cdot 1 + 2 \cdot 0 = 1$

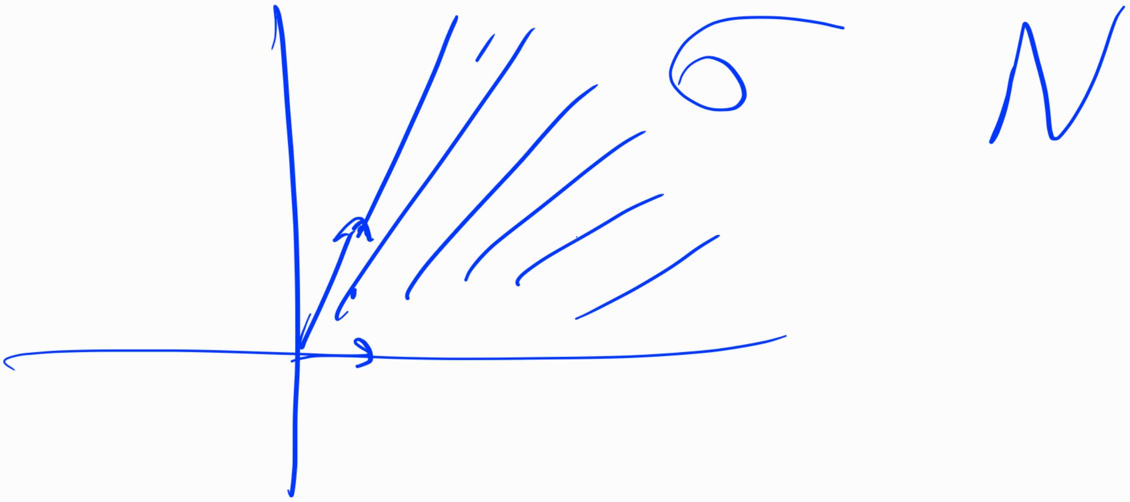
c) $u \cdot v = 1$

$u \rightsquigarrow x$, $v \rightsquigarrow x^{-1}$, $w \rightsquigarrow y$

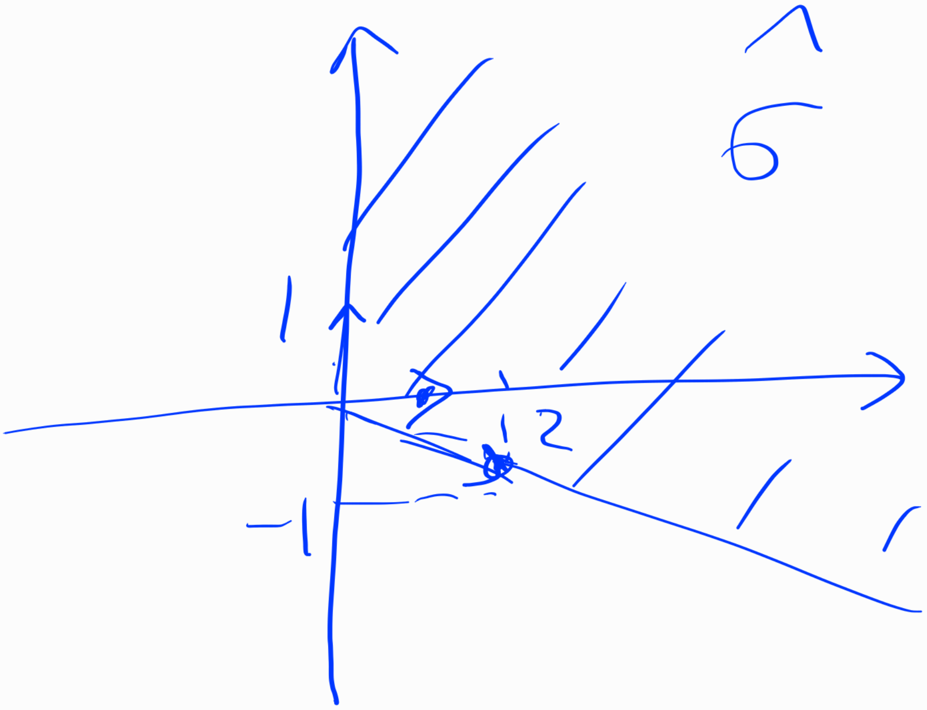
$\mathbb{C}[\hat{\sigma} \cap M] = \mathbb{C}[x, x^{-1}, y]$

in $\operatorname{Spec}(\mathbb{C}[x, x^{-1}, y]) \cong \mathbb{C}^* \times \mathbb{C}$

②



$\hat{\sigma}$ M



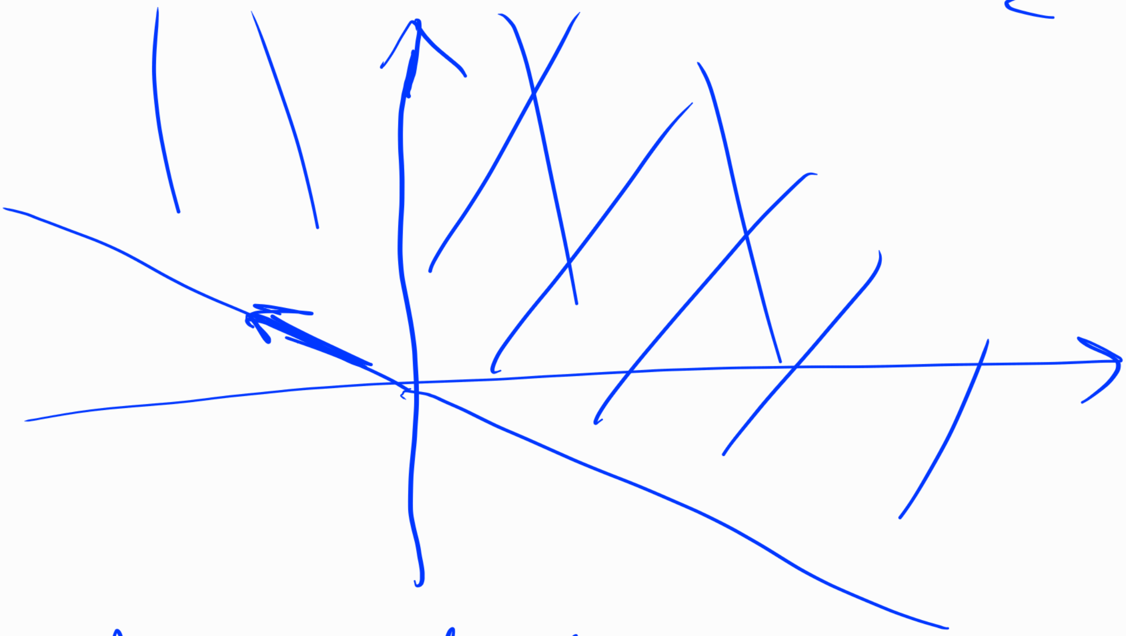
$$\begin{cases} m_1 \cdot 1 + m_2 \cdot 0 \geq 0 \\ m_1 \cdot 1 + m_2 \cdot 2 \geq 0 \end{cases}$$

$m_1 \geq 0$

$$\hat{\sigma} \cap M = \langle \underset{\vec{u}}{(2,-1)}, \underset{\vec{v}}{(0,1)}, \underset{\vec{w}}{(1,0)} \rangle$$

$$\vec{u} + \vec{w} = 2\vec{v}$$

$\overset{1}{x} \cdot \overset{1}{z} = \overset{1}{y}^2 \quad \hat{\sigma}_1 \quad \hat{\sigma}_2$



We need to allow $(-2,1) \rightarrow \frac{1}{x}$

$\mathbb{C}[x,y,z]/(xz-y^2)$

Localize by $\{1, x, x^2, \dots\}$

Def. R domain.
Then R is integrally closed if $\forall \frac{a}{b} \in Q(R)$ if $\frac{a}{b}$ is integral over R, then $\frac{a}{b} \in R$.

Thm. If R is integrally closed and S is a multiplicative system in R, then R_S is integrally closed.

Proof. $R \subset R_S \subset Q(R)$

$$\begin{matrix} Q(R) \\ \shortparallel \\ Q(R_S) \end{matrix}$$

Suppose $\frac{a}{b} \in Q(R_S)$ is

$\overset{\shortparallel}{Q}(R)$

integral over R_S:

$$\left(\frac{a}{b}\right)^n + \underset{\underset{R_S}{\cap}}{\left(\frac{c_{n-1}}{s_{n-1}}\right)}\left(\frac{a}{b}\right)^{n-1} + \dots + \left(\frac{c_0}{s_0}\right) = 0$$

Call $\frac{a}{b} = X$

$$\underline{X^n + \left(\frac{c_{n-1}}{s_{n-1}}\right)X^{n-1} + \dots + \left(\frac{c_0}{s_0}\right) = 0} \quad \cdot s^n$$

$$\underline{s = s_{n-1} \cdot \dots \cdot s_0}$$

$y = Xs$

$$s^{n-1}X^{n-1} = y^{n-1}$$

$\swarrow$

$$y^n + \underbrace{\left(\frac{c_{n-1}}{s_{n-1}}\right) \cdot s}_{\in R} y^{n-1} + \dots + \underbrace{\left(\frac{c_0}{s_0}\right) s^n}_{\in R} = 0$$

So because R is integrally closed, $y \in R$.

$y = xs$, $x = \frac{y}{s} \in R_s$ $\square$

<u>Thm.</u> Suppose R is a domain. Suppose $\forall$ maximal ideal $m \subseteq R$ R_m is integrally closed. Then R is integrally closed.

(**Reminder** R_m is a localization of R w.r.t. $S = R \setminus m$)

Proof. Suppose $\frac{a}{b} \in Q(R)$ that is integral over R.

$$\left(\frac{a}{b}\right)^n + \underline{c_{n-1}}\left(\frac{a}{b}\right)^{n-1} + \ldots + \underline{c_1}\left(\frac{a}{b}\right) + c_0 = 0$$

So $\forall m$ max ideal in R

$\left(\frac{a}{b}\right)$ is in R_m

Consider $I \subset R$

$I = \{ r \in R \mid r \cdot (\frac{a}{b}) \in R \}$

1) I is an ideal in R

2) If $(\frac{a}{b}) \notin R$, then

$I \neq R$.

(If $I = R$, then $1 \in I$
so $\frac{a}{b} \in R$)

If $\frac{a}{b} \notin R$, $I \subsetneq R$

So $\exists m$ maximal in R

$I \subseteq m$

$\frac{a}{b} \in R_m$:

$\frac{a}{b} = \frac{x}{s}$ $s \notin m$

$$s\frac{a}{b} = \begin{array}{c} x \\ \cap \\ R \end{array}$$

So $s \in I$ Contradicts $s \notin m$

($I \subseteq m$) $\square$

Def: R is a comm. ring with 1. Krull dimension of R is the $\sup(n)$ s.t. $\exists$ prime ideals

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

_Ex._ 1) Suppose R is a domain with Krull dimension 0. Then R is a field

2) $\mathbb{C} \oplus \mathbb{C} = \{(x_1, x_2)\}$

$(1,0)\cdot(0,1) = 0$

3) $\mathbb{C}[x]/(x^2)$

$\underline{[x]\cdot[x] = 0}$

The only prime ideal is $([x])$

4) Krull dimension $(\mathbb{Z})$ is 1

— // — $\mathbb{C}[x]$ is 1

— // — $F[x]$ (field) is 1

5) Krull dimension of $\mathbb{C}[x_1, \dots, x_n]$ is n

$$(x_1, x_2, \dots, x_n) \supsetneq (x_1, x_2, \dots x_{n-1})$$

pt line

$$\supsetneq \dots \supsetneq (x_1) \supsetneq (0)$$

$\boxed{x_1 = 0}$

$$\mathbb{C}[x_1 \dots x_n] / \underline{(x_1, \dots, x_k)}$$

$$\cong \underline{\mathbb{C}[x_{k+1}, \dots, x_n]}$$

domain

<u>Thm.</u> Suppose $R = \mathbb{C}[x_1 \dots x_n] / I$

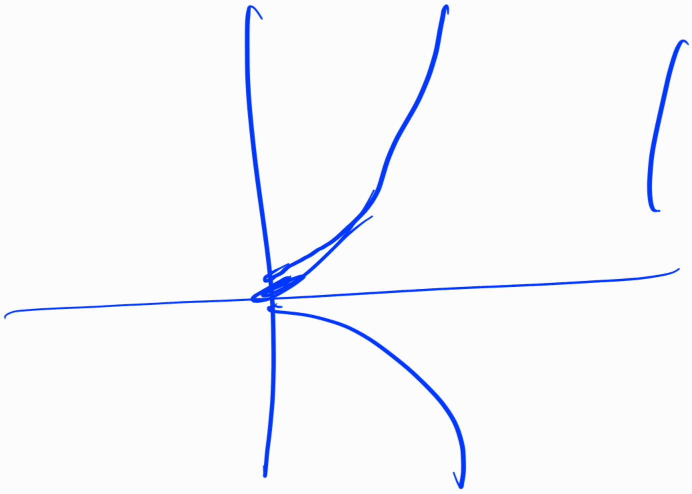
is a domain.
Then Krull dimension of R is the transcendence degree of $Q(R)$ over $\mathbb{C}$.

Key part of the proof

R, $f \in R$ not a zero divisor.

Then Krull dim of $R/(f)$ is (Krull dim. of R) -1

Ex. $\mathbb{C}[x,y]/(y^2-x^3)=R$



("cusp")

$\dim(R)=1$

Not normal:

$$\left(\frac{[y]}{[x]}\right)^2=[x]$$

$$\begin{tikzcd} \frac{[y]}{[x]} \arrow[d, phantom, "\cap" rotate=90] & {[x]} \arrow[d, phantom, "\cap" rotate=90] \\ Q(R) & R \end{tikzcd}$$

"Non-normal toric variety"

$S = \{0, 2, 3, 4, \ldots\}$

$\mathbb{C}[S] = \mathbb{C}[x, y]/(y^2 - x^3)$