$\mathbb{C}[x_1, \dots x_n]/I$

<u>Def.</u> Transcendence degree

$F_1 \subseteq F_2$

$\operatorname{trdeg}(F_2/F_1)$ is the largest number of F_1-algebraically independent element in F_2

$y_1, \dots, y_k \in F_2$

$F_1[t_1, \dots, t_k] \xrightarrow{f} F_2$

$t_i \mapsto y_i$

$\{y_1 \dots y_k\}$ are alg. independent iff f is injective.

<u>Milne</u> Field Theory Chapter 9

$F_1 \subseteq F_2$

tr. basis of F_2 over F_1

is $\{y_1, \dots y_k\} \subset F_2$

s.t. 1) $\{y_1 \dots y_k\}$ alg. indep.

2) $F_2 : F_1(y_1 \dots y_k)$

is algebraic.

Noether Normalization Theorem.

$\mathbb{C}[x_1 \dots x_n]/I = R$

Then $\exists\ y_1, \dots, y_d \in R$

s.t.

$\mathbb{C}[y_1 \cdots y_d] \hookrightarrow \mathbb{C}[x_1 \dots x_n]/I$

($y_1 \cdots y_d$ are alg. ind. / $\mathbb{C}$)

and R is integral

over $\mathbb{C}[y_1 \cdots y_d]$.

Ex. $\mathbb{C}[x_1, x_2]/(x_2^3 - x_1^3 - x_1)$

$y_1 = x_1$

or $y_1 = x_2$

...

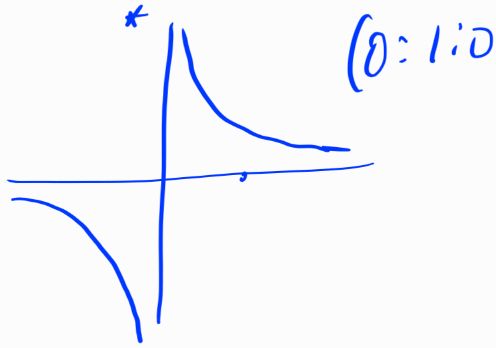
Ex. $\mathbb{C}[x_1, x_2]/(x_1 x_2 - 1)$

$y_1 = x_1$ does not work

x_2 is not integral over x_1

$y_1 = x_1 + x_2$ works

$x_2 = x_1^{-1}$



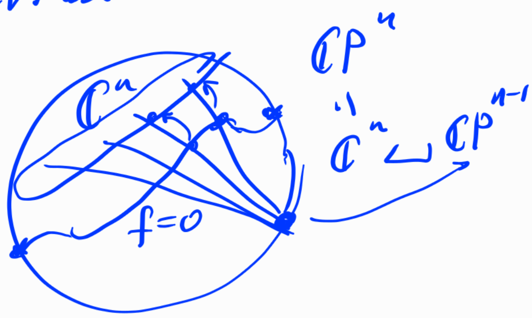
Proof. 1) We will prove this, choosing y_i to be $\mathbb{C}$-linear combin. of $x_1, \dots x_n$

2) Use induction on n.

If $R = \mathbb{C}[x_1, \dots x_n]$, there is nothing to prove.

If not, then $\exists\, f \in I$, $f \neq 0$, where $R = \mathbb{C}[x_1, \dots x_n]/I$

We will show that after some linear change of variables, f becomes monic in one of the variables.



$f \in \mathbb{C}[x_1 \dots x_n]$

Look at the part of f of the highest total degree

$$f(x_1,x_2,x_3) = 2x_1x_3 + \underbrace{x_1^3 + x_2x_3^2}_{\text{total degree } 3}$$

Homogenizing: $(x_1 : x_2 : x_3 : x_4)$

$2x_1x_3\underline{x_4} + x_1^3 + x_2x_3^2$
a homog. equation for $f=0$
in $\mathbb{C}P^4$. At $x_4 = 0$
only $x_1^3 + x_2x_3^2$ matters.

Take some
$(x_1 \dots x_n) \in \mathbb{C}^n \setminus \{(0,\dots 0)\}$
s.t. $f_h(x_1 \dots x_n) \neq 0$.

highest degree part of f

$(x_1 : x_2 : \dots : x_n : 0)$
Make linear substitution
so that in my new
variables this point
this is $(1,0,0,\dots 0)$

$y_1 = 1,\ y_2 = \dots = y_n = 0.$

<u>Claim</u>. The equation $f=0$
is now monic in y_1

$f = f_h + \underline{\text{lower degree}}$

$f_h(1,\underline{0,\dots,0}) \neq 0$

$f_h = \sum a_? y_1^{k_1} \cdots y_n^{k_n}$

$\underline{k_1 + \dots + k_n} = m$ — max total degree.

f_h has a term $\underset{\neq 0}{a} y_1^m$

$f = a y_1^m +$ lower degrees in y_1

So y_1 is integral
over $y_2 \dots y_m$.
By induction, applied
to $\mathbb{C}[y_2 \dots y_m]/\mathcal{Y} = R'$

$$\begin{array}{c} \cap \\ R \end{array}$$

$\exists$ $z_1 \dots z_d$ linear
in $y_2 \dots y_m$
s.t. $\mathbb{C}[z_1 \dots z_d] \hookrightarrow R' \subset R$
s.t. R' is integral
over $\mathbb{C}[z_1 \dots z_d]$
Then $\forall i = 2, \dots n$
y_i is integral ove
$\mathbb{C}[z_1 \dots z_d]$.
And y_1 is integral
over $y_2, \dots y_n$
So All y_i $i = 1, \dots n$
are integral over
$\mathbb{C}[z_1, \dots z_d]$ $\square$

Thm. This d is unique.
Idea of a proof.
R integral over
$[y_1 \dots y_d]$, R is f. gen.
as $\mathbb{C}$ algebra.
This means that
R is a f.g. module
over $\mathbb{C}[y_1 \dots y_d]$
Set of generators
$\{x_1^{i_1} \dots x_n^{i_n}\}$ $i_j \in \{0, 1, \dots, k_j - 1\}$

Suppose $\{z_1, \dots, z_m\} \subset R$

where $m > d$.

We look for polynomials in $z_1 \dots z_m$ of degree $\le D$ in each variable.

We write $F(z_1 \dots z_m)$ is

"coordinates" $\{x_1^{i_1} \dots x_n^{i_n}\}$

coordinates are in $\mathbb{C}[y_1 \dots y_d]$

$I \subseteq \mathbb{C}[x_1 \dots x_n]$

In general, I is the intersection of f.many "primary" ideals

Milne Commutative Algebra

If $\mathrm{rad}(I) = I$, then these primary ideals are prime ideals