Discrete valuation rings

<u>Def.</u> A ring R is called DVR if

1) The only prime ideals are $\{0\}$ and $p = (x)$

2) Every non-zero ideal is (x^n) for some n.

<u>Ex.</u> p prime in $mSpec(\mathbb{Z})$

$$R = \left\{ \frac{n}{m} \,\middle|\, p \nmid m \right\} \subset \mathbb{Q}$$

(p); Every non-zero ideal in R is (p^n)

$n \geq 0$

Discrete Valuations

(additively)

Def. $v: F^* \twoheadrightarrow \mathbb{Z}$

field

is called a discrete valuation on F if

1) v is a group homomorphism

$(v(1) = 0, \quad v(xy) = v(x) + v(y))$

2) $v(x+y) \geq \min(v(x), v(y))$

Ex. p-adic valuation

on $\mathbb{Q}$

$$v_p\left(\frac{n}{m}\right) = \mathrm{ord}_p(n) - \mathrm{ord}_p(m)$$

To a discrete valuation on F we can associate

$$R = \{x \in F \mid v(x) \geq 0\}$$

We have the prime ideal (x) where x is any $x \in F$ $v(x) = 1$

$$x_1 \cdot x_2 = xy$$

$$x_1 \cdot x_2 \in (x)$$

$$v(x_1) + v(x_2) = 1 + val(y)$$

$$\geq 1$$

$$\begin{cases} v(x_1) \geq 1 \\ v(x_2) \geq 1 \end{cases}$$

$$v\left(\frac{x_1}{x}\right) \geq 0 \quad \frac{x_1}{x} \in R$$

$$x_1 \in (x)$$

$\forall y \in R \quad y \neq 0$

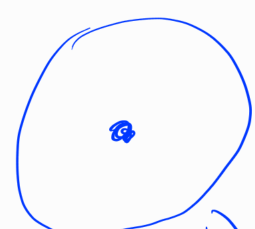
$v(y) = n \in \mathbb{Z}_{\geq 0}$

$$y = \underbrace{\left(\frac{y}{x^n}\right)}_{\text{unit in } R} \cdot x^n$$

$$(y) = (x^n)$$

field

DVR



<u>Def.</u> An element in DVR with valuation 1 is called a uniformizer

Krull dimension of DVR

$(0) \neq P$ is 1.

DVR is integrally closed

DVR is Noetherian

DVR is a <u>local</u> ring

<u>Def.</u> A ring (comm, with 1) is local if it has 1 maximal ideal.

<u>Thm.</u> Suppose R is Noetherian local ring that is a domain, has Krull dimension 1, and is integrally closed. Then R is DVR.

Proof. (by Milne, section 20)

Suppose $c \in R$, $c \neq 0$

For $b \in R$, $b \notin (c)$

$I_b = \{a \in R \mid ab \in (c)\}$

I_b is $\mathrm{Ann}(\bar{b})$ in the R-module $R/(c)$

Consider b such that I_b is not R and is maximal out of all I_b.

Claim. I_b is prime.

Proof of claim.

Suppose $x_1 \cdot x_2 \in I_b$

$$\underbrace{x_1 \cdot x_2 \cdot b}_{} \in (c)$$

Case 1 $x_2 \cdot b \in (c)$

Then $x_2 \in I_b$

Case 2. $x_2 \cdot b \notin (c)$

Then $I_{x_2 \cdot b} \supseteq I_b$

$$\begin{cases} x_1 \in I_{x_2 b} \\ I_b \text{ is maximal} \\ I_{x_2 \cdot b} \neq R \end{cases} \Rightarrow x_1 \in I_b$$

$\square$

R, $P \ \{0\}$ prime ideal

Claim: $P = \left(\frac{c}{b}\right)$

Proof of claim.

1) $\frac{b}{c} \notin R \qquad b \notin (c)$

2) $p \cdot b \subset (c)$

$\overset{\shortparallel}{I_b} \qquad p \cdot \frac{b}{c} \subset R$

ideal

If $p \cdot \frac{b}{c} \neq R$

then $p \cdot \left(\frac{b}{c}\right) \subseteq p$

$\left(\frac{b}{c}\right)$ acts on p

finitely generated

P is a module over R

$\left(\frac{b}{c}\right)$ acts on P

$P = \langle x_1, \dots, x_n \rangle$

$\left(\frac{b}{c}\right)$-action is determined by $\left(\frac{b}{c}\right)x_1, \dots \left(\frac{b}{c}\right)x_n$ as linear comb. of $x_1 \dots x_n$ with coeff. in R,

$\left(\frac{b}{c}\right)$ is "multiplication by a matrix" M with entries in R.

$\left(\frac{b}{c}\right)$ is a zero the char.

poly of M, which

is monic.

So $\left(\frac{b}{c}\right)$ is integral over R

So $\left(\frac{b}{c}\right) \in R$.

But $\frac{b}{c} \notin R$.

So $p\left(\frac{b}{c}\right) = R$

$$\boxed{p = \left(\frac{c}{b}\right)}$$

So p is generated by

one element.

Call it π (the

uniformizer).

We want to show:

$\forall a \in R$

$a = \pi^n \cdot u$
$\qquad\qquad |$
$\qquad\quad$ unit.

$a, \frac{a}{\pi}, \frac{a}{\pi^2}, \ldots$

$(a) \subseteq \left(\frac{a}{\pi}\right) \subseteq \left(\frac{a}{\pi^2}\right) \subseteq \ldots$ as long as they are in R, R is Noetherian

If $\left(\frac{a}{\pi^k}\right) = \left(\frac{a}{\pi^{k+1}}\right)$

$\frac{\left(\frac{a}{\pi^k}\right)}{\pi}$ is a unit,

then

$\frac{a}{\pi^{k+1}} = y \cdot \frac{a}{\pi^k} \Rightarrow 1 = y\pi$

So $\exists$ max k s.t.

$$\frac{a}{\pi^k} \in R,$$

$$\frac{a}{\pi^{k+1}} \notin R$$

In ~~fact~~ if $\frac{a}{\pi^k}$ is not a unit, then $\frac{a}{\pi^k} \in P$

So $\frac{a}{\pi^k} \in (\pi)$, $\frac{a}{\pi^{k+1}} \in R$

contradiction. $\square$

Next: Compactness

$\mathbb{C}^*$, $\mathbb{C}$ not compact in $\mathbb{C}$ topology

$\mathbb{CP}^1$ compact in $\mathbb{C}$-topology