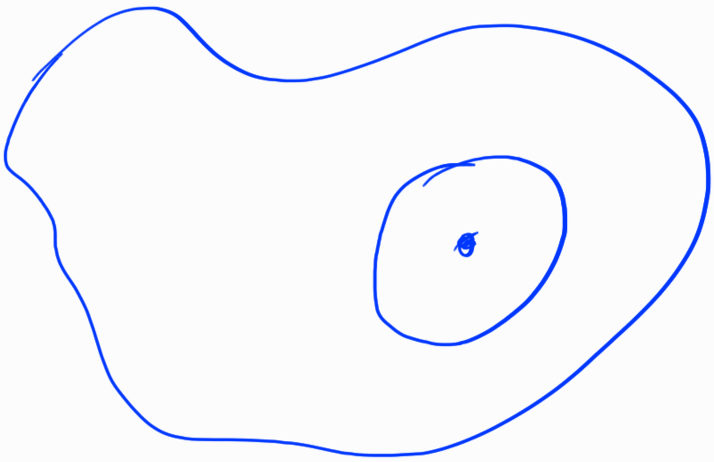
$\mathbb{C}^*$, $\mathbb{C}$ $\mathbb{CP}^1$



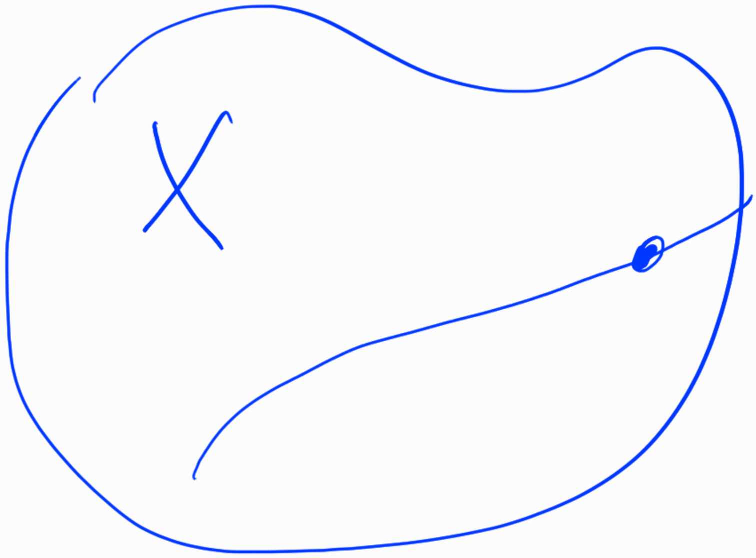
$R \subset Q(R)$

R is a DVR

Condition. For every DVR R, for any map

$$\begin{tikzcd} \operatorname{Spec}(Q(R)) \arrow[r] \arrow[d] & X \\ \operatorname{Spec}(R) \arrow[ur] & \end{tikzcd}$$

$\exists !$ $\operatorname{Spec}(R)$



Thm. If X is a subvariety of a projective space $\mathbb{CP}^n$ given by a system of homogeneous polynomials then X is complete.

$$\begin{tikzcd} X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

In dimension 1

Compact 1-dim. complex manifolds (smooth)

$\wr\wr$

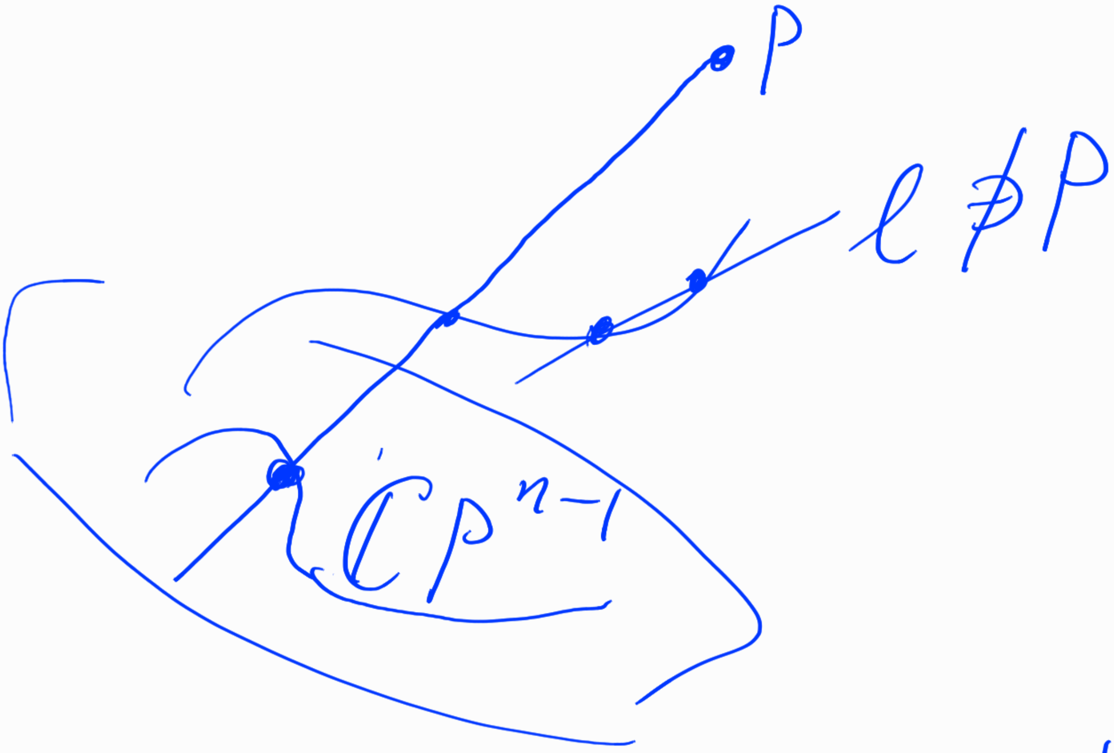
Projective algebraic curves that are smooth $/\mathbb{C}$

$\wr\wr$

Transcendence degree 1 extensions of $\mathbb{C}$.

<u>Thm.</u> Every compact complex curve can be embedded into $\mathbb{C}P^3$

① $X \hookrightarrow \mathbb{C}P^4$



$$\mathbb{CP}^n \setminus P \longrightarrow \mathbb{CP}^{n-1}$$

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d

g genus

$$2g-2 = d^2 - 3d$$

$$X \subseteq \mathbb{CP}^n$$

$\dim X = 1$

$$(X \cap \mathbb{C}^n) \subset \mathbb{CP}^n$$

domain, prime

$$X = \mathrm{mSpec}\,(\overbrace{\mathbb{C}[x_1 \ldots x_n]/I}^{R=})$$

$Q(R)$ has transc. deg 1 over $\mathbb{C}$

GAGA paper (Serre, 56)

$F \supset \mathbb{C}$ tr. deg $(F) = 1$

$\Downarrow$

t $\quad \underbrace{F \supset F(t)}_{\text{finite}} \supset \mathbb{C}$

$$\begin{array}{ccc} X & & F \\ \downarrow f & & \uparrow \\ \mathbb{C}P^1 & & \mathbb{C}(t) \end{array}$$

degree $=$ degree$(F : \mathbb{C}(t))$

$= \# f^{-1}(a)$

for generic a.

<u>Def.</u> Gonality of X

is the smallest possible

such d.

$d=1$ $\quad \mathbb{C}P^1$

$d=2$ $\quad$ hypergeometric X

In higher dimensions:

$\exists$ smooth compact complex surfaces that are not algebraic

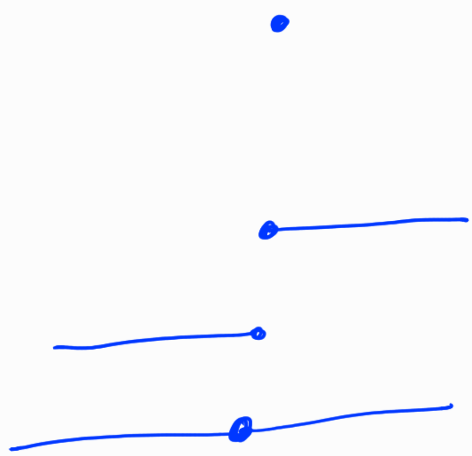
Ex. $\mathbb{C}^2 / L$

$L \cong \mathbb{Z}^4$

Barth-Peters-van-de-Ven-Hulek (?)

X toric variety defined by a fan Σ in N

Dim 1.



$\mathbb{C}^*$

$\mathbb{C}$

$\mathbb{C}$

$\mathbb{CP}^1$

Thm. Toric variety X_Σ is complete

(= compact as $\mathbb{C}$-space)

iff $\bigcup_{\sigma \in \Sigma} \sigma = N \otimes_{\mathbb{Z}} \mathbb{R}$

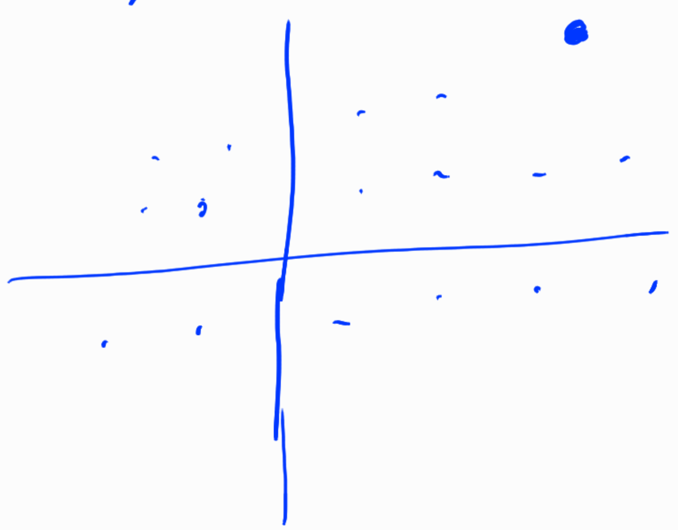
Idea of the proof.

Easy direction.

If $\bigcup_{\sigma \in \Sigma} \sigma \neq N_{\mathbb{R}}$

then X_Σ is not complete.

$N \simeq \mathbb{Z}^d$



Take $\vec{n} \in N$ which is not in any of our cones $\sigma \in \Sigma$

$\vec{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$

$x_1, x_2, \dots, x_n$

$(\mathbb{C}^*)^n = m\,\mathrm{spec}(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$

$$\begin{array}{ccc} \mathbb{C}^* & \xrightarrow{f} & (\mathbb{C}^*)^n \\ \psi & & \psi \\ t & \longmapsto & (t^{n_1}, t^{n_2}, \dots, t^{n_d}) \end{array}$$

$a \in \mathbb{Z} \setminus \{0\}$

<u>Note</u>

$(n_1, \dots, n_d) \rightsquigarrow (an_1, \dots an_d)$

$(t^{an_1}, \dots, t^{an_d})$

$\parallel$

$((t^a)^{n_1}, \dots, (t^a)^{n_d})$

$$\begin{tikzcd} \mathbb{C}^* \arrow[r] \arrow[d, "t \to t^a"'] & (\mathbb{C}^*)^d \\ \mathbb{C}^* \arrow[ur] & \end{tikzcd}$$

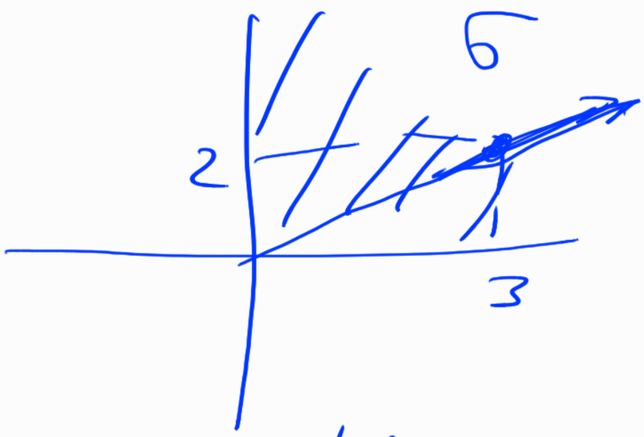
We cannot compactify $f(\mathbb{C}^*)$ in $(\mathbb{C}^*)^d$

(as $t \to \infty$ or 0

$t^{n_i} \to \infty$ or 0

if $n_i \neq 0$)

Ex. $d=2$



$\hat{\sigma}$

$\mathbb{C}^2 \ (x_1, x_2)$

$$t \longrightarrow (x_1^3, x_2^2)$$

$$t \in \mathbb{C}^*$$

$$\mathbb{C}^* \subset \mathbb{C}$$

extra point is $t=0$

$$\begin{tikzcd} \mathbb{C}^* \arrow[r] & \mathbb{C}^2 \\ \mathbb{C} \arrow[ur] & \end{tikzcd}$$

$$0 \longmapsto (0,0)$$

If $\vec{n}$ is not in any σ then as $t \to 0$ there is no way to complete this in

$$X_\sigma = \operatorname{mSpec} \mathbb{C}[\sigma^\wedge \cap M]$$

$\exists$ monomial

$$x_1^{m_1} \cdots x_d^{m_d} = g$$

s.t. $n_1 m_1 + \dots + n_d m_d < 0$

This is a function g on X_σ

Composing g with

$$t \mapsto (x_1^{n_1}, \dots, x_d^{n_d})$$

we get

$$t \mapsto t^{n_1 m_1 + \dots + n_d m_d}$$

as $t \to 0$, this goes to ∞.