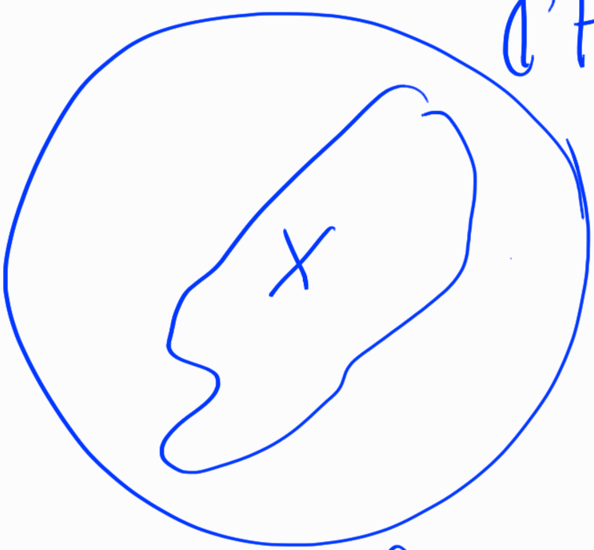
More algebraic geometry

X algebraic variety
(usually projective
or, more generally, complete
varieties)

X irreducible

$\mathbb{CP}^N \supset \mathbb{C}^N$



$X \cap \mathbb{C}^N$
affine variety

$R = \mathbb{C}[x_1, \dots, x^N]/I$

polynomials that
vanish at X

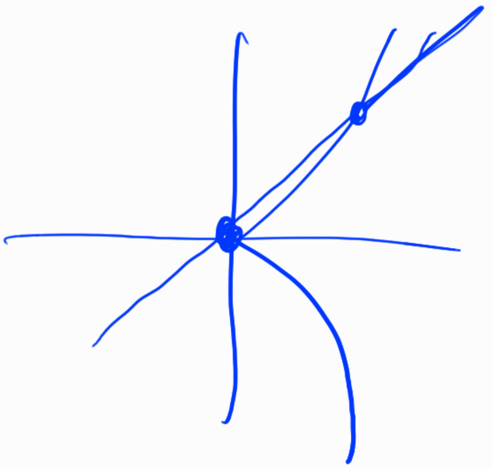
I prime ideal
(X is irreducible)

R is a domain

$Q(R) = K(X)$ is a field

We call elements of $K(X)$ rational functions on X

Ex. $X = \operatorname{mSpec} \mathbb{C}[x,y]/(y^2-x^3)$



$K(X)$ has tr. deg 1 over $\mathbb{C}$

Claim $K(X) \cong \mathbb{C}(t)$

where $t = \frac{[y]}{[x]}$

Proof. $\begin{cases} y^2 = x^3 \\ y = tx \end{cases}$ $\quad t^2x^2 = x^3$

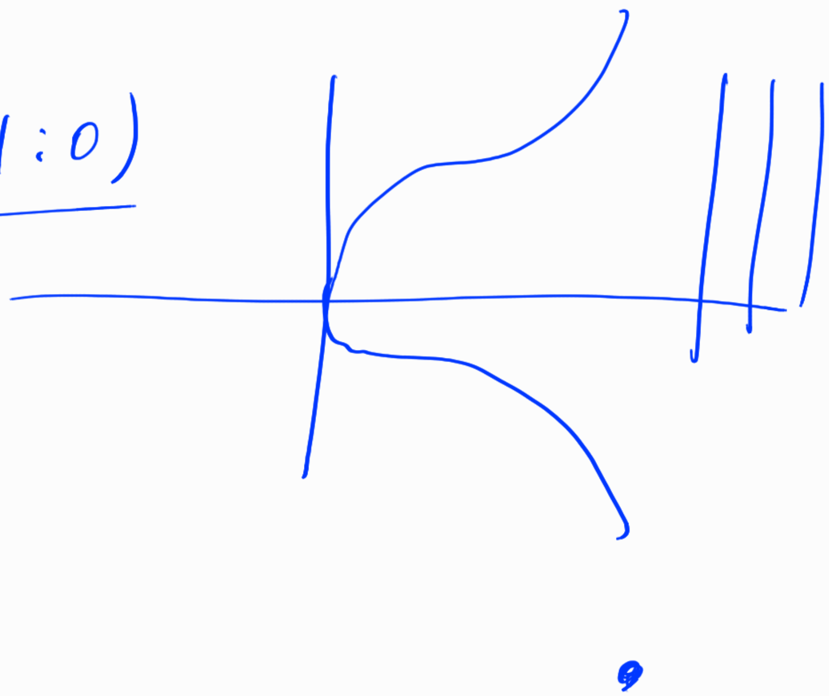
fixed $\quad x = 0$

$\begin{cases} x = t^2 \\ y = t^3 \end{cases}$

Ex. $X = \mathrm{mspec}(\mathbb{C}[x,y]/(y^2 - x^3 - x))$

X is an "elliptic curve"

$(0:1:0)$



$\dim(X) = 1$ $\quad X$ curve, smooth

Def. A divisor on X is a formal finite sum of points of X with integer coefficients.

Def. If D is a divisor on X then $\deg(D)$ is the sum of its coefficients

Def. Suppose $f \in K(X)$
$\ne$
0

Then $\mathrm{div}(f) = \sum_{p \in X} v_p(f) \cdot \underbrace{[p]}$

order of zero/pole
of f at p

X normal, so
the localization at p
of the ring R is DVR
v_p is that discrete valuation.

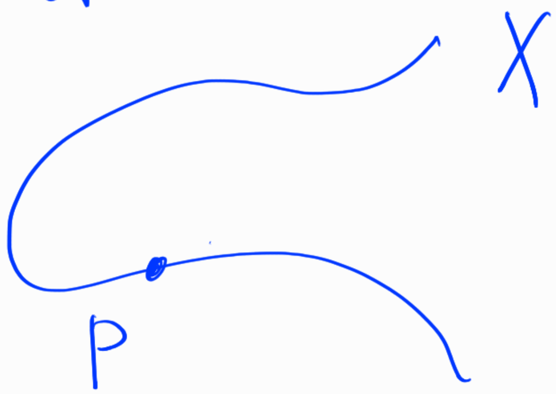
K tr. deg 1 over $\mathbb{C}$
$\ni$
f

$K = Q(R)$

$f = \frac{f_1}{f_2}$ $\qquad f_1, f_2 \in R$
$\qquad f_2 \neq 0, f_1 \neq 0$

$P \quad R_P \subseteq K$

"all rational functions on X that are defined at P"



$m \subset R_P$

$\shortparallel$

$\{ f$ that are 0 at $P \}$

If we take $\pi \in m \setminus m^2$

then every $f \in R_P$ is uniquely $\underbrace{u \cdot \pi^k}_{\text{unit } u \in R_P \setminus m}$ $\quad k \geq 0$

V_P can be extended to $Q(R)$

by $V_P\left(\frac{f_1}{f_2}\right) = V_P(f_1) - V_P(f_2)$

Q. What is the structure of the (group of all divisors on X) / (group of all $\mathrm{div}(f)$) ?

$$\mathrm{div}(f_1 \cdot f_2) = \mathrm{div}(f_1) + \mathrm{div}(f_2)$$

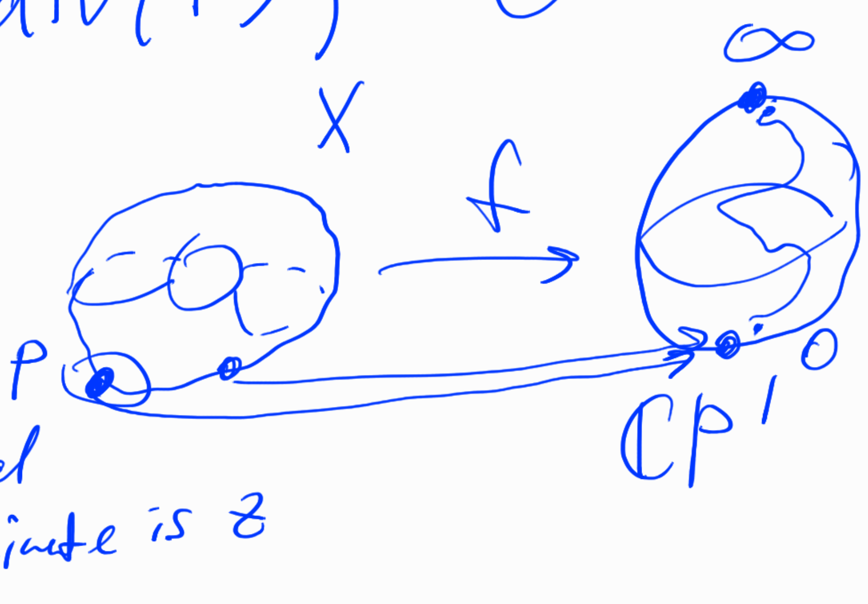
This group is called divisor class group.

Thm. Suppose X is smooth, projective curve.

Then $\forall f \in K$, $f \neq 0$

$$\deg(\mathrm{div}(f)) = 0$$

Proof.



locally, $f: z \to cz^k + \dots$, $c \neq 0$

$\mathrm{div}(f) = \dots + k[p] + \dots$

$$k = \# f^{-1}\left((q \approx 0) \cap U_p\right)$$

$$cz^k = \varepsilon$$

Algebraic Number Theory

$\mathbb{Z} \subset \mathbb{Q}$	$\mathbb{C}[t] \subset \mathbb{C}(t)$
$\vdots \quad \vdots$	$\vdots \quad \vdots$
$R \subset K$	$R \quad K$

fractional ideals in R

$I \subset K$ s.t. $\exists\, a \in K$, $a \neq 0$

aI is an ideal in R

Fractional ideals form a group, free abelian group on prime ideal

$$Cl = \underline{(\text{fractional ideals})} / \underline{(\text{principal ideals})}$$

K

$\downarrow$

(f)

$\not\ni$

0

Thm. Cl is finite

$\deg(\mathrm{div}(f)) = 0$

$\deg : \mathrm{Div}(X)/\sim \longrightarrow \mathbb{Z}$

$D_1 \sim D_2 \iff D_1 - D_2 = \mathrm{div}(f)$

Kernel of this map is an abelian variety (a compact algebraic group) of dim g = genus(X)

which is called

Jacobian (X)

<u>Ex.</u> $g=0$ $\mathbb{CP}^1$

$\ker = \{0\}$ $\quad Jac(\mathbb{CP}^1)=\{0\}$

<u>Ex.</u> $g=1$ E elliptic curve

$Jac(E) \cong E$

Main Ingredient:

Riemann-Roch Theorem

X smooth, complete $\quad D$ divisor on X

$$H^0(D) = \{0\} \cup \{ \underset{K\setminus\{0\}}{f} \mid D + \mathrm{div}(f) \geq 0 \}$$

<u>Ex.</u> $D = 2[p_1] + 3[p_2]$

$H^0(D) =$ "all meromorphic

functions on X that have pole of order at most 2 at p_1, at most 3 at p_2 and no other poles

$H^0(0) = \mathbb{C}$

$H^0(D)$ finite-dim. vector space / $\mathbb{C}$

$h^0(D) = \dim H^0(D)$

If $D_1 \sim D_2$ $(D_1 - D_2 = \mathrm{div}(f))$

then $h^0(D_1) = h^0(D_2)$

<u>RR Thm.</u>

H^0 space

h^0 dim

$$h^0(D) - \underline{h^0(K-D)} = \deg(D) + 1 - g$$

g genus

K "the divisor of a 1-form"

$g = h^0(K)$