More on divisors

$d=1$ X smooth (complete) curve $/\mathbb{C}$

A divisor on X is a formal finite sum of points of X with $\mathbb{Z}$ coefficients

A divisor of a function:

$f \in K = \mathbb{C}(X)$

$f \neq 0$ (rational or meromorphic function)

$\mathrm{div}(f)$ is the divisor of zeroes, poles of f.

$$\operatorname{div}(f_1 \cdot f_2) = \operatorname{div}(f_1) + \operatorname{div}(f_2)$$

<u>Def.</u> $D_1 \sim D_2$

(linearly or rationally equivalent) if $D_1 - D_2 = \operatorname{div}(f)$

<u>Def.</u> Divisor class group of X: group of divisors modulo linear equivalence

---

<u>Def.</u> D is a divisor on X

$$H^0(D) = \{0\} \cup \{\underset{K \setminus \{0\}}{\underset{\cap}{f}} \mid D + \operatorname{div}(f) \geq 0\}$$

finite-dimensional vector space over $\mathbb{C}$ <u>if X is complete</u>

$$h^0(D) = \dim H^0(D)$$

<u>Lemma.</u>

If $D_1 \sim D_2$ then

$$h^0(D_1) = h^0(D_2)$$

Proof. Suppose $\underline{D_1 - D_2 = \operatorname{div}(f)}$

$\forall g \in K \quad g \in H^0(D_2)$
$\;\;\neq 0$

Then $\operatorname{div}(g) + D_2 \geq 0$

$$\Updownarrow$$

$$\operatorname{div}(g) - \operatorname{div}(f) + \underbrace{\operatorname{div}(f) + D_2}_{D_1} \geq 0$$

$$\Updownarrow$$

$$\operatorname{div}\left(\frac{g}{f}\right) + D_1 \geq 0$$

$$\Updownarrow$$

$$\frac{g}{f} \in H^0(D_1)$$

$$H^0(D_1) \underset{\cdot\left(\frac{1}{f}\right)}{\overset{\cdot f}{\rightleftarrows}} H^0(D_2)$$

So $h^0(D_1) = h^0(D_2)$

Riemann-Roch Theorem

$$h^0(D) - h^0(K-D) = \deg D + 1 - g$$

K — canonical class

"divisors of differential forms"

$f(z)\,dz$ z is a local parameter.

Algebraically

<u>Kahler Differentials</u>

$f_1 \cdot d f_2$ $f_1, f_2 \in K$

Locally at a point P on X we want to write Kahler differential

s.t. f_2 to be a uniformizer

at p: $\quad d\nu(f_2) = [p] + \ldots$

$$\frac{df_2}{d\pi} \quad \text{a function in } K(X)$$

<u>Ex.</u> $X: \quad y^2 - x^3 - x = 0$

$$\frac{dy}{dx}? \qquad 2y\frac{dy}{dx} - 3x^2 - 1 = 0$$

$$\frac{dy}{dx} = \frac{3x^2 - 1}{2y} \in K(X)$$

To define a divisor

of $f_1 df_2$:

at p s.t. f_2 is a

uniformizer at p,

take $\operatorname{ord}_p(f_1)$.

Ex. $X = \mathbb{CP}^1 \supset \mathbb{C}$

variable $x = \frac{X_1}{X_2}$

On $\mathbb{CP}^1$ $(X_1 : X_2)$ $\quad X$ $\quad ''$

If $X_2 \neq 0$ $(X_1 : X_2) = \left(\frac{X_1}{X_2} : 1\right)$

Consider $1 \cdot dx$ $\qquad \pi = 0$

At the point $(1:0)$

x is not a uniformizer

$\frac{1}{x} = \pi$ is uniformizer

$x = \frac{1}{\pi}$ $\qquad 1 \cdot dx = -\frac{1}{\pi^2} d\pi$

divisor of $(1 \cdot dx)$ is $-2[\infty]$

$-2[(1:0)]$

degree is -2

Ex. $\mathbb{CP}^1$

$\omega = (x-1) \cdot d(x^3 - 3x)$

$$\omega = 3(x-1)(x^2-1)\,dx \qquad dx = d(x-1)$$
$$= 3(x-1)^2(x+1)\,dx \qquad \text{''} \; d(x+1)$$

$$x = \frac{1}{\pi} \quad \omega = 3\left(\frac{1}{\pi}-1\right)^2\left(\frac{1}{\pi}+1\right)\underbrace{\left(-\frac{1}{\pi^2}\right)d\pi}$$

divisor of ω is:

$$\underline{2\cdot[1]} + \underline{1\cdot[-1]} - 5\cdot[\infty]$$

degree: $2+1-5 = -2$

In general, if ω_1, ω_2 are two Kahler differentials, their divisors are linearly equivalent

$\frac{\omega_1}{\omega_2}$ is a rational function

<u>Def.</u> K is the divisor class of differential forms.

$g = h^0(K)$

In general:

$\deg(K) = 2g-2$

$\mathbb{CP}^1 \quad g=0 \quad \deg(K) = -2$

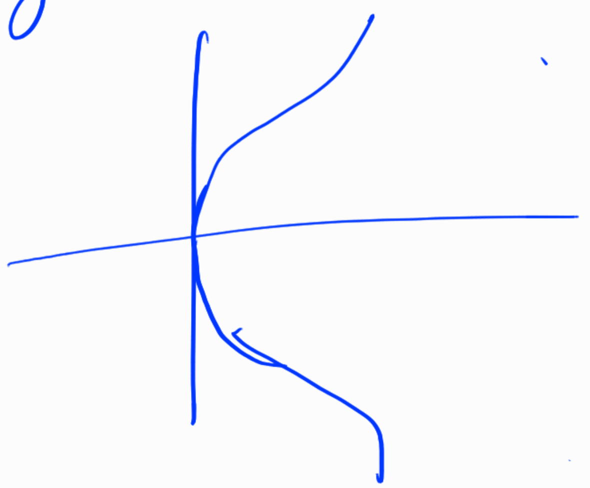
$E \quad y^2 = x^3 + x$

$g=1 \quad \deg(K) = 2\cdot 1 - 2 = 0$

$h^0(K) = g = 1$

$\omega = \frac{dx}{y}$ has divisor 0.

$y=0$ at $(x,y) = (0,0), (i,0), (-i,0)$



locally at $(0,0)$ x^3+x

$y^2 = x(x^2+1)$

y is a uniformizer

$2y\,dy = (3x^2+1)\,dx$

$$\frac{dx}{y} = \underbrace{\frac{2}{3x^2+1}}_{\neq \text{ at } x=0}\,dy$$

$0, \infty$

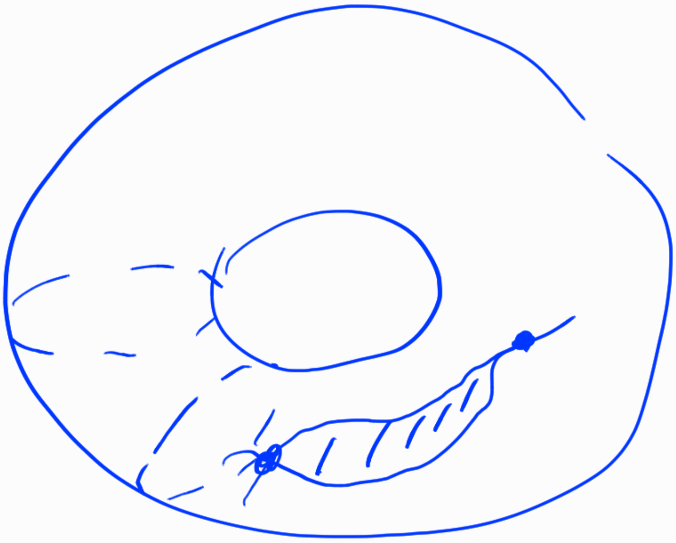
The origin of a lot of mathematics:

elliptic integrals

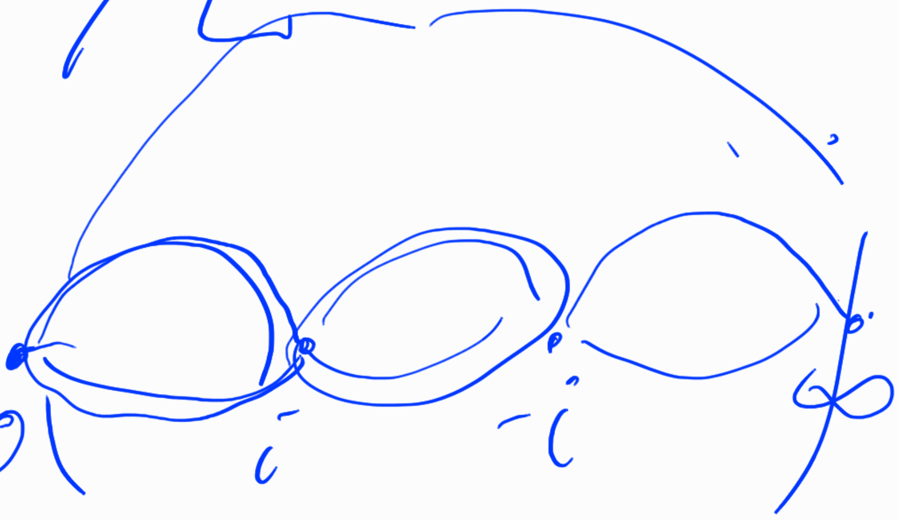
$$\int \frac{dx}{\sqrt{x^3+x}}$$

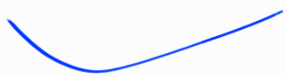
Jacobi

$+ \ldots$



$\mathbb{C}/L$





$$E \xrightarrow{x} \mathbb{CP}^1$$

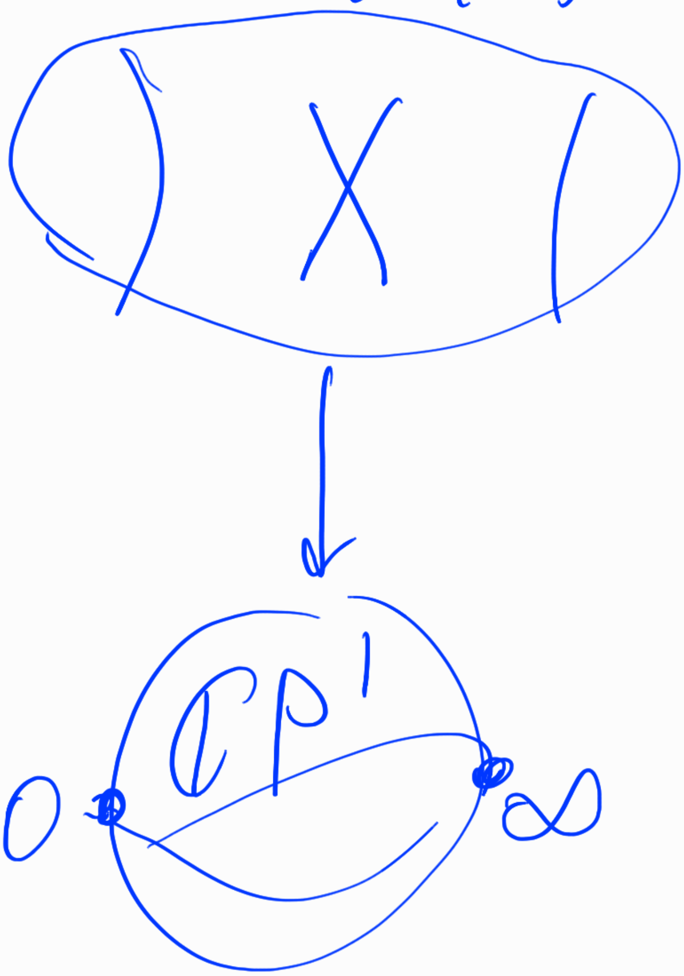
Higher dimensions

$\dim(X) = d.$

X normal (complete)

Def. A Weil divisor on X is a finite formal sum of codimension 1 subvarieties of X.

$$f \in K(X)$$
$$\neq 0 \rightsquigarrow \mathrm{div}(f)$$



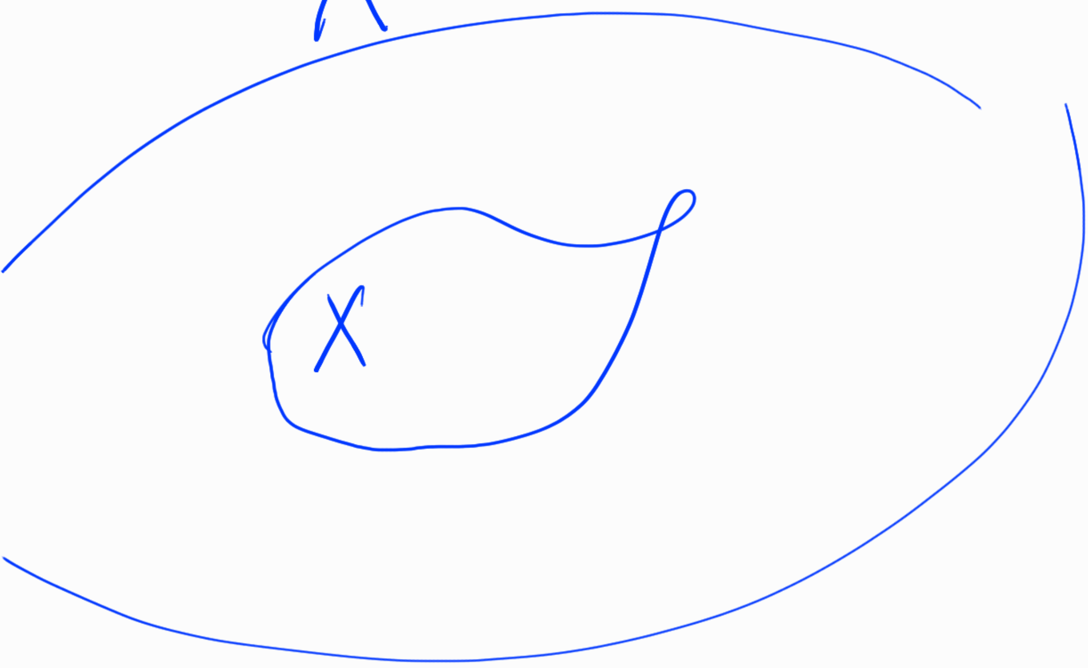
Divisor class group:

All divisors / divisors of functions

(modulo linear equivalence)

Very important general example

$$X \subset \mathbb{CP}^N$$



A generic hyperplane

H of $\mathbb{C}P^N$ is given

by homogeneous equation

$$c_1 x_1 + c_2 x_2 + \dots + c_{N+1} x_{N+1} = 0$$

$$H \cap X = D \quad \text{divisor on } X$$

(effective)

$$D \geq 0$$

Any two such H are

linearly equivalent:

$$H \quad c_1 x_1 + \dots + c_{N+1} x_{N+1} = 0$$

$$H' \quad c_1' x_1 + \dots + c_{N+1}' x_{N+1} = 0$$

$$f = \frac{c_1 x_1 + \dots + c_{N+1} x_{N+1}}{c_1' x_1 + \dots + c_{N+1}' x_{N+1}}$$

restricted to X is a

rational function on X,

s.t. $\mathrm{div}(f) = D - D'$

The system of all such divisors on X is called a linear system.

In particular, given any divisor D on X, the complete linear system

$$|D| = \{ D' \mid D' \sim D \}$$

$$H^0(D) = \{ f \mid \underbrace{\mathrm{div}(f) + D}_{= D'} \geq 0 \} \cup \{0\}$$