$H^0(D)$

Ex. $\mathbb{C} = m\mathrm{Spec}(\mathbb{C}[x])$

$D = 0$

$$H^0(D) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; \begin{array}{l} \forall a \in \mathbb{C} \\ q(a) \neq 0 \end{array} \right\}$$

$$= \{p(x)\} = \mathbb{C}[x]$$

$D = 2[\pi] - 3[0]$

$$H^0(D) = \left\{ \frac{x^3 \cdot p(x)}{(x-\pi)^2} \;\middle|\; p(x) \in \mathbb{C}[x] \right\}$$

Ex. $X = \mathbb{C}P^1$

$D = 2[\pi] - 3[0] + 4[\infty]$

$$H^0(D) = \left\{ \frac{x^3 \cdot p(x)}{(x-\pi)^2} \;\middle|\; 3 + \deg p - 2 \leq 4 \right\}$$

$$H^0(D) = \{ f \mid D + \underline{\mathrm{div}(f)} \geq 0 \}, \quad f \neq 0$$

$$H^0(D) = \left\{ \frac{x^3 p(x)}{(x-a)^2} \;\middle|\; \underline{\deg p \leq 3} \right\}$$

$$h^0(D) = \dim H^0(D) = 4$$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

a_0, a_1, a_2, a_3 any in $\mathbb{C}$

$$\deg(D) = 3$$

<u>R-R.</u>

$$\underbrace{h^0(D)}_{=4} - \underbrace{h^0(K-D)}_{=0} = 3 + 1 - \overbrace{g}^{=0}$$

$$\deg K = -2 \qquad \deg(K-D) = -5 < 0$$

If $\deg(D_1) < 0$,

then $H^0(D_1) = \{0\}$

$$H^0(D_1) = \{0\} \cup \{f \mid \underline{\mathrm{div}(f) + D_1 \geq 0}\}$$

$$\deg(\mathrm{div}(f) + D_1) = \deg D_1 < 0$$

For $\mathbb{CP}^1$

$$h^0(D) = \begin{cases} \deg(D)+1, & \deg D \geq 0 \\ 0, & \deg D < 0 \end{cases}$$

$$D \sim (\deg D) \cdot [\infty]$$

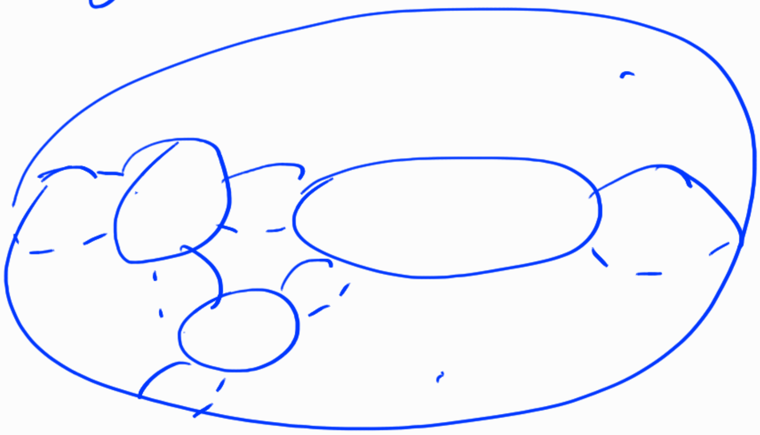
$$H^0(D) = \{p(x) \in \mathbb{C}[x] \mid \deg p \leq \deg D\}$$

<u>Riemann-Roch Inequality</u>

$$h^0(D) \geq \deg(D) + 1 - g$$

And "=" if $\deg(D)$ is large enough $(\deg D > 2g-2)$

Say, $g=3$



Corollary

If $\deg(D)>0$, then D is ample: for large enough $n \geq 0$ $H^0(nD)$ can be used to embed X into some projective space.

Ex. $P \in X$ one point

$nD = n[P]$

$$H^0(nD) = \{f \in K(X) \mid \operatorname{div}(f) + n[P] \geq 0\}$$

Take a basis of $H^0(nD)$

$\{f_1, \ldots, f_d\}$

$$X \longrightarrow \mathbb{C}P^{d-1}$$
$$x \longmapsto (f_1(x) : f_2(x) : \ldots : f_d(x))$$

If at some $x \in X$

all $f \in H^0(nD)$ are 0,

then $H^0(nD) = \underline{H^0(nD - [x])}$

($\supseteq$ always)

Higher dimensions

X dim d

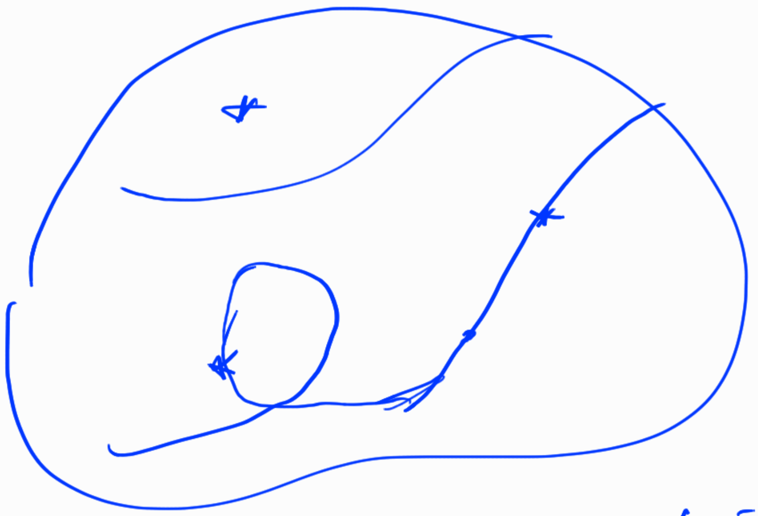
Divisors are formal sums of dim $(d-1)$

irreducible subvarieties.

div(f) = ... zeroes and poles

(We assume that X is normal, which implies that it is "smooth in codimension 1")

$\underline{\dim X = 2}$ X normal



These are $\underline{\text{Weil divisors}}$

There are also $\underline{\text{Cartier divisors}}$

A Weil divisor D is a Cartier divisor if

at every point $p \in X$
it is equal to a divisor
of a function.

$$X \ni p$$
$$\cup\!\!\mid$$
$$m\mathrm{Spec}(R)$$

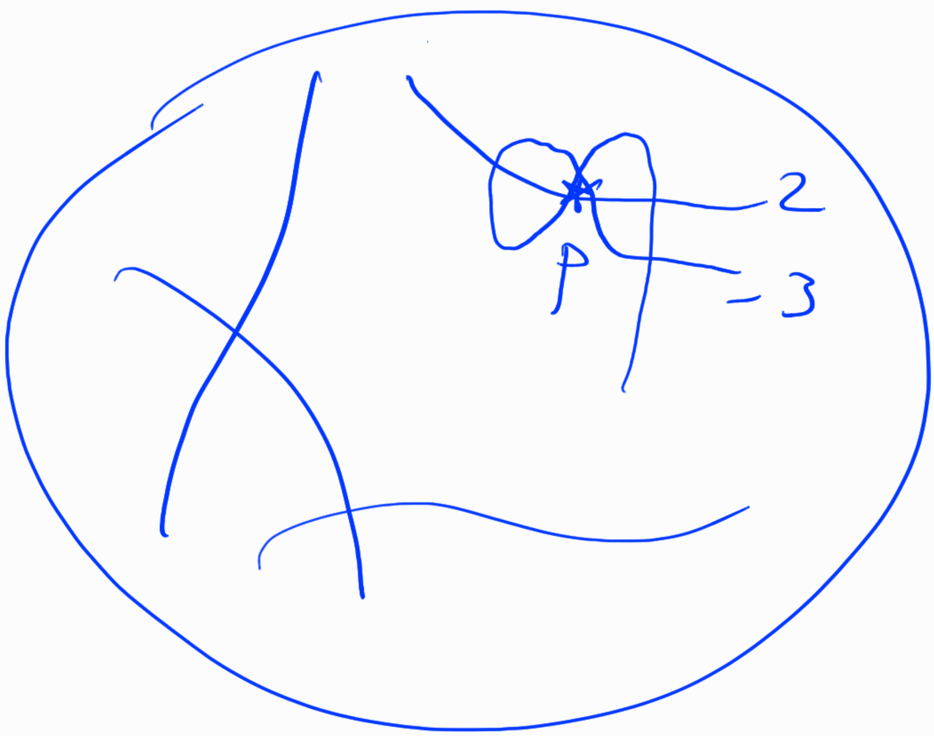
$$R \subseteq R_p$$

R_p localization of R
at p

Divisor on X, restricted to
some neighborhood of p
may become a divisor of
a function.

$\mathrm{Pic}(X) \longrightarrow \underline{\mathrm{Pic}(R_p)}$

Cartier divisors are Weil divisors in the kernels of these maps for all p.



Go back to toric varieties.

M monomials

$N = \mathrm{Hom}(M, \mathbb{Z})$

Σ fan in N

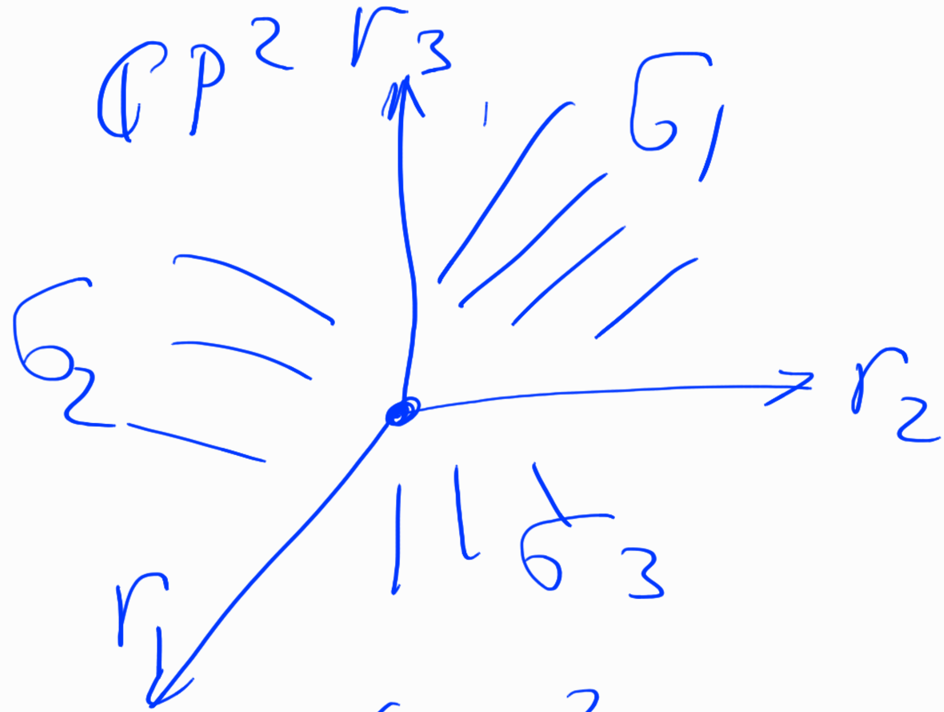
a collection of strictly convex rational closed polyhedral cones $\{\sigma\}$ in N.

We glue X from $m\mathrm{Spec}(\mathbb{C}[\hat{\sigma} \cap M])$

Every strictly convex rational closed polyhedral cone σ is generated by its rays (1-dimensional faces of σ)

Ex. $\mathbb{CP}^2$

N



$\{\vec{0}\}$ —— $(\mathbb{C}^*)^2$

r_1, r_2, r_3

$\sigma_1, \sigma_2, \sigma_3$

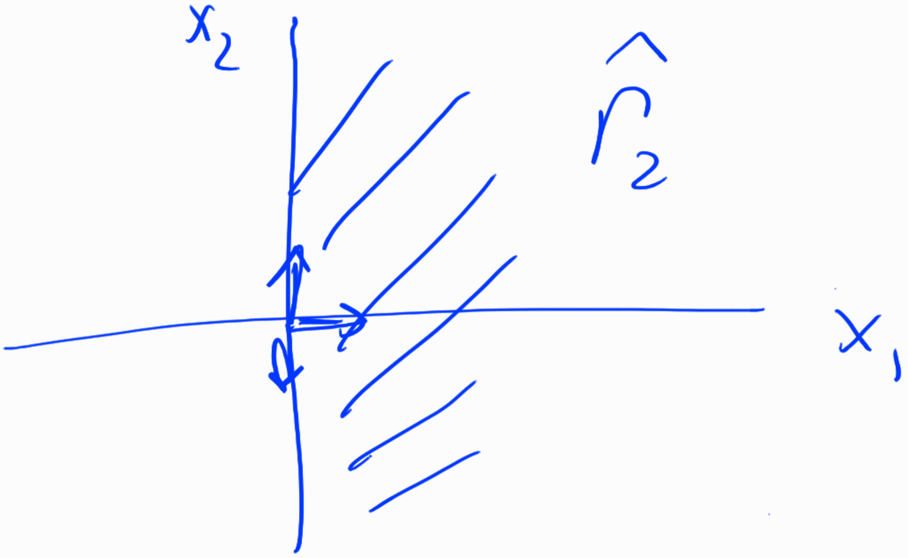
Take r_2

$r_2 = \langle (1,0) \rangle$

$\hat{r}_2 = \{(m_1, m_2) \in M$ s.t. $m_1 \cdot 1 + m_2 \cdot 0 \geq 0\}$

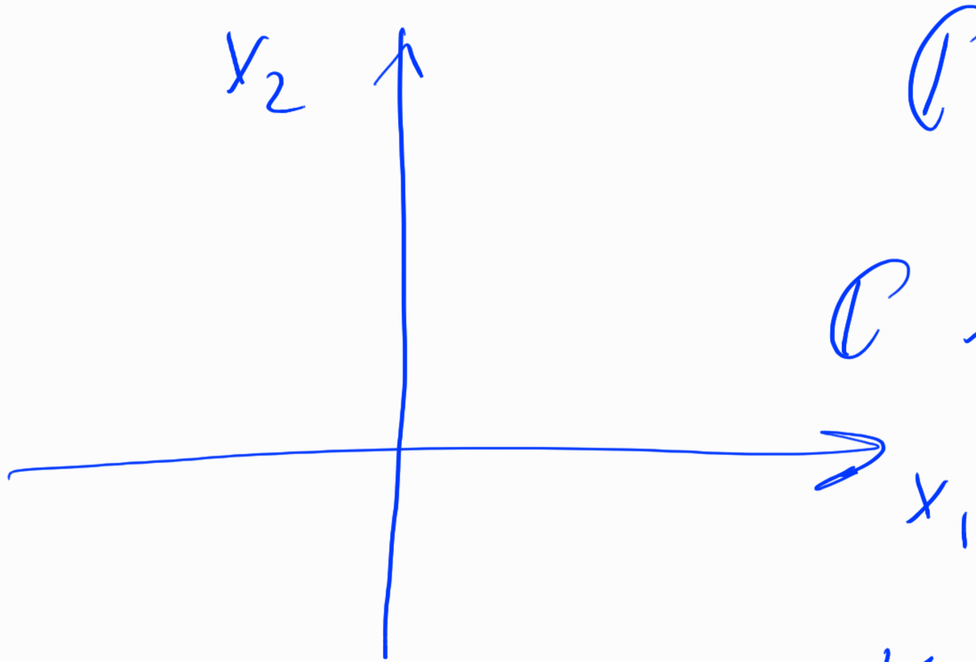
$m_1 \geq 0$

M



$$\mathbb{C}(\hat{r}_2 \cap M) = \mathbb{C}[x_1, x_2, x_2^{-1}]$$

$$\mathrm{mSpec}(\mathbb{C}[x_1, x_2, x_2^{-1}]) \cong \mathbb{C} \times \mathbb{C}^*$$



$\mathbb{C}^* \times \mathbb{C}^*$

$\cap$

$\mathbb{C} \times \mathbb{C}^*$

$\mathbb{C} \times \mathbb{C}^*$ "filling in the $x_1 = 0$ axis"

We can use this "chart" to define the order of zero/pole of a rational function on $x_1 = 0$ curve. This is the valuation that corresponds to $(1,0) \in N$

f is a Laurent polynomial in x_1, x_2.

$$\mathrm{val}_{(1,0)} f = \inf(1\cdot m_1 + 0\cdot m_2)$$ over all monomials $a x_1^{m_1} x_2^{m_2}$ in f with $a \neq 0$.

$\mathrm{val}(x_1) = 1$

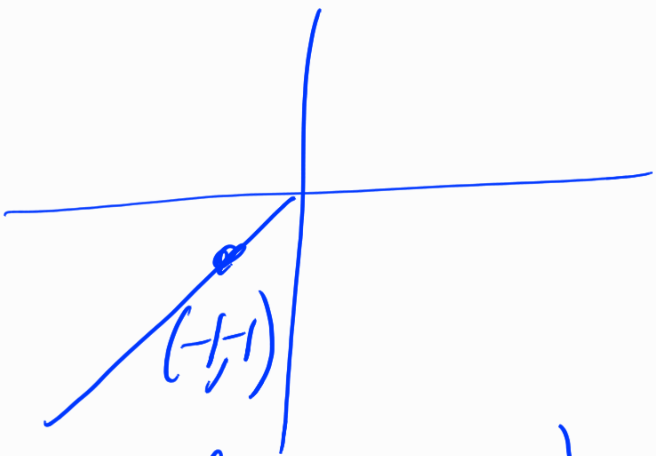
$\mathrm{val}(x_2) = 0$

<u>Three rays</u> r_1, r_2, r_3

$\mathbb{CP}^2$ has 3 lines:

$x_1 = 0$, $x_2 = 0$, line at ∞

$(x_1 : x_2 : 1)$



valuation: (−total degree)

$X_1^2 X_2^3$ has a pole of order 5 at the line at ∞.

$(-1)\cdot 2 + (-1)\cdot 3 = -5$

Important.

$\operatorname{Pic}((\mathbb{C}^*)^n) = \{0\}$

(Unique factorisation in Laurent polynomials)