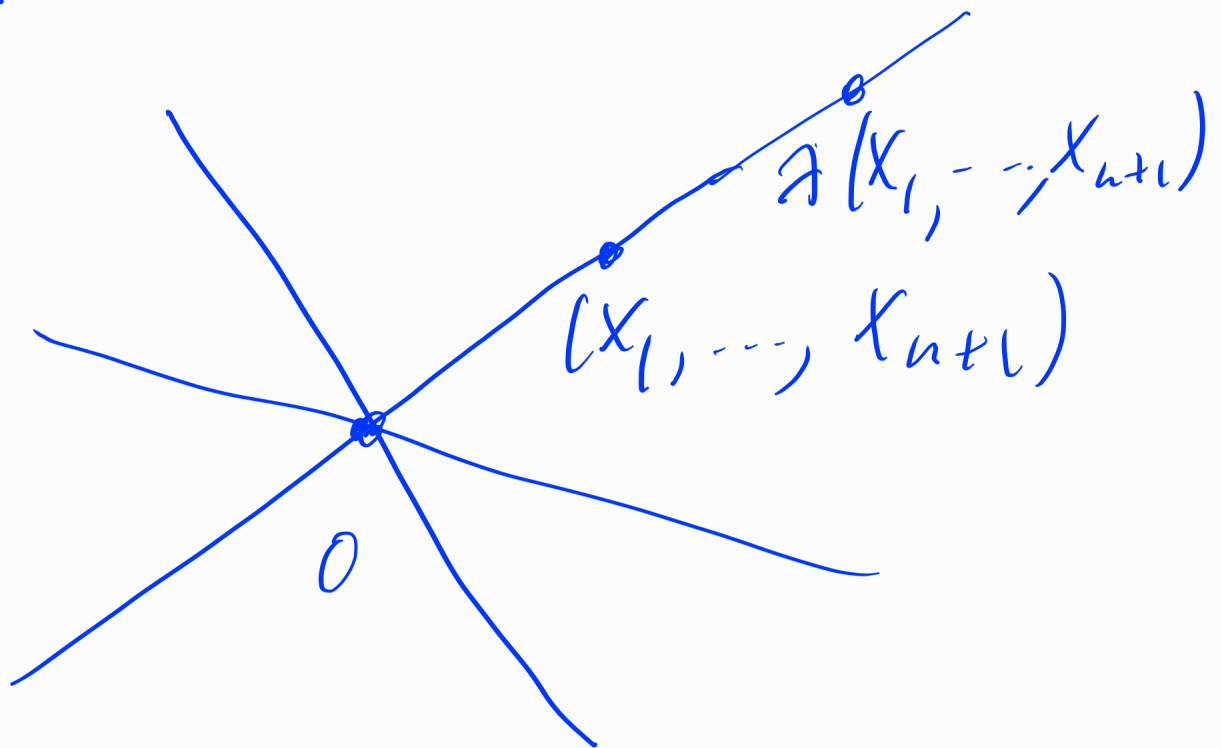


Projective spaces

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{\vec{0}\}) / \sim$$

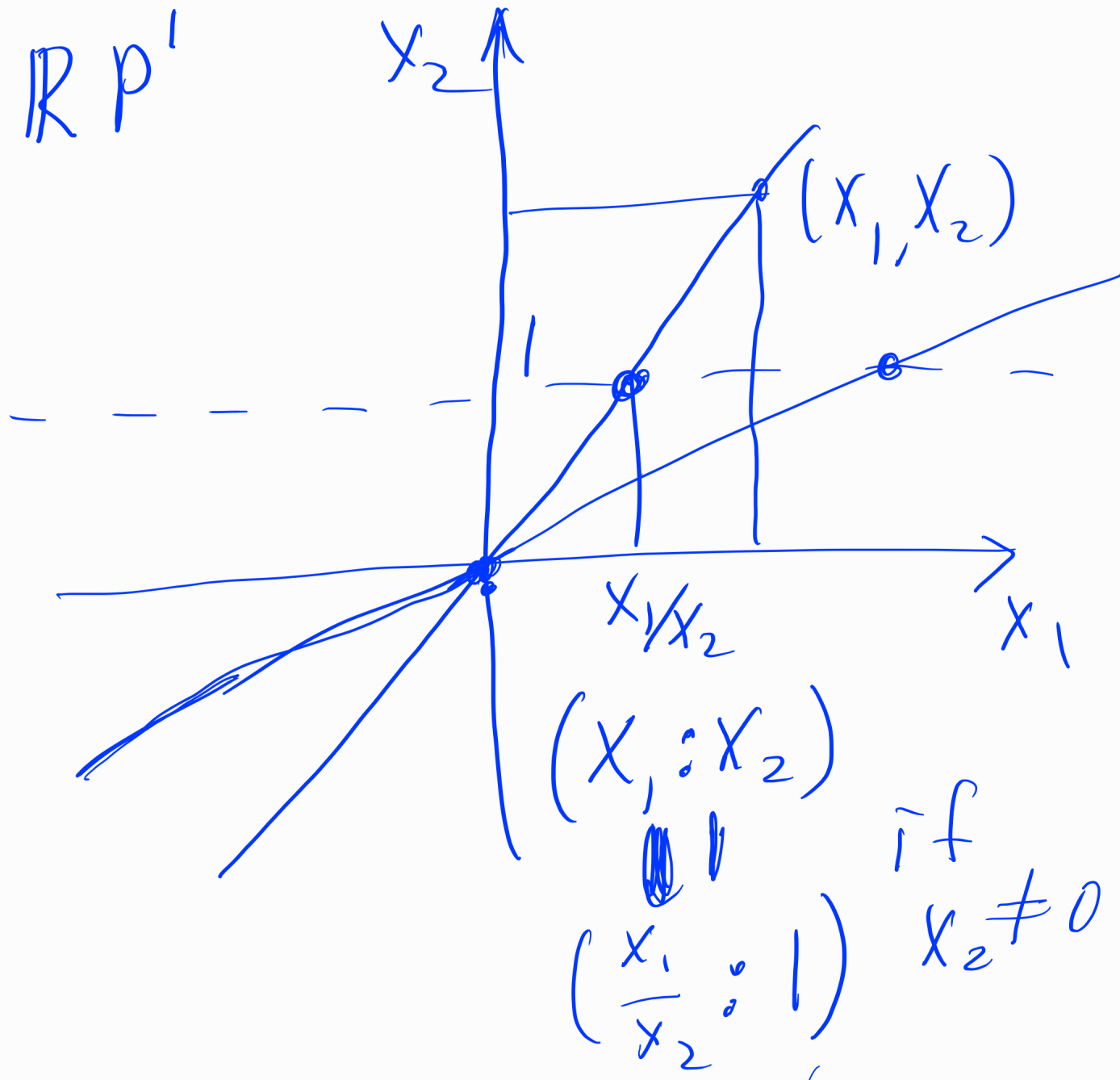
$$(x_1, \dots, x_{n+1}) \sim (\lambda x_1, \dots, \lambda x_{n+1})$$

$$\lambda \in \mathbb{R} \setminus \{0\}$$



$$(x_1 : x_2 : \dots : x_{n+1})$$

$\mathbb{R}P^1$



$$\mathbb{P}^n \supset \mathbb{A}^n \text{ (affine space)}$$

$$\mathbb{R}\mathbb{P}^n \supset \mathbb{R}^n$$

$$(x_1 : x_2 : \dots : x_{n+1})$$

Case 1 $x_{n+1} \neq 0$

Case 2 $x_{n+1} = 0$

In case 1

$$(x_1 : x_2 : \dots : x_{n+1})$$

$$\left(\frac{x_1}{x_{n+1}} : \frac{x_2}{x_{n+1}} : \dots : \frac{x_n}{x_{n+1}} : 1 \right)$$



$\mathbb{R}^n$

In particular:

$$\mathbb{RP}^1 = \mathbb{R} \sqcup \underline{\{(1:0)\}}$$

↑
case 1

$$(x_1 : x_2)$$

$$\text{If } x_2 = 0$$

Any n

$$\mathbb{RP}^n = \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}$$

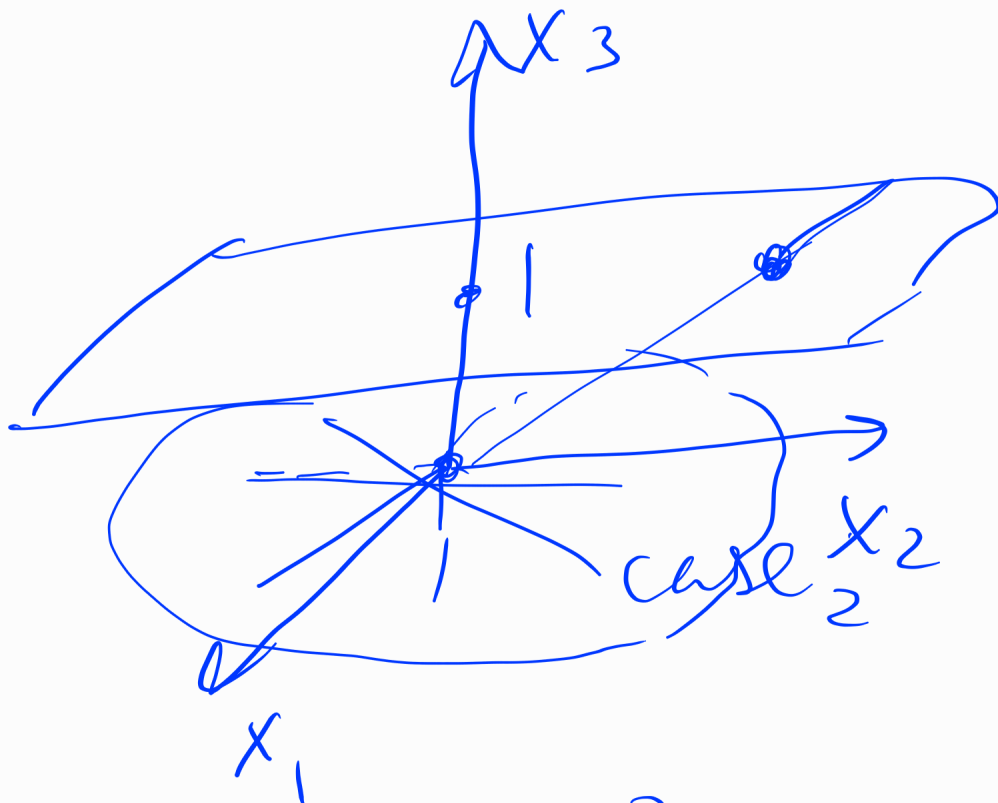
↑
case 1

↑
case 2

$$x_{n+1} \neq 0$$

$$x_{n+1} = 0$$

$$(x_1 : x_2 : \dots : x_n : 0) \quad \mathbb{RP}^{n-1}$$



$$\mathbb{RP}^2 = \mathbb{R}^2 \sqcup \mathbb{RP}^1_{\star}$$

$$ax + by = c$$

*

$$x = \frac{x_1}{x_3}, y = \frac{x_2}{x_3}$$

$$ax_1 + bx_2 = cx_3$$

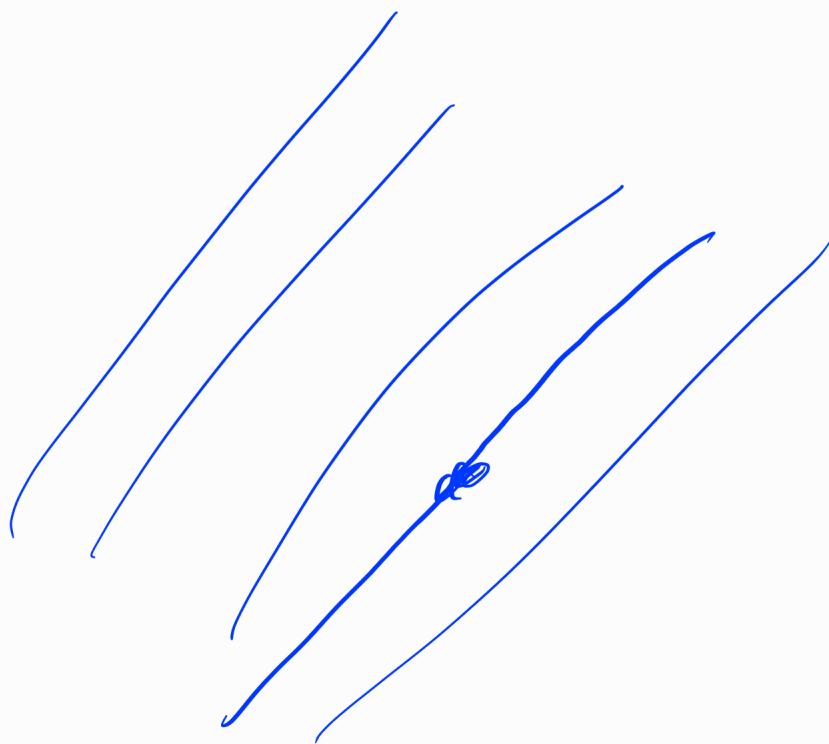
" $(x_1 : x_2 : x_3)$
line at ∞ ": $x_3 = 0$

Ex. $x + 2y = 5$ $\angle$

$\angle \cap (x_3 = 0) \text{ --- ?}$

$(2 : -1 : 0) \quad (-1 : \frac{1}{2} : 0)$

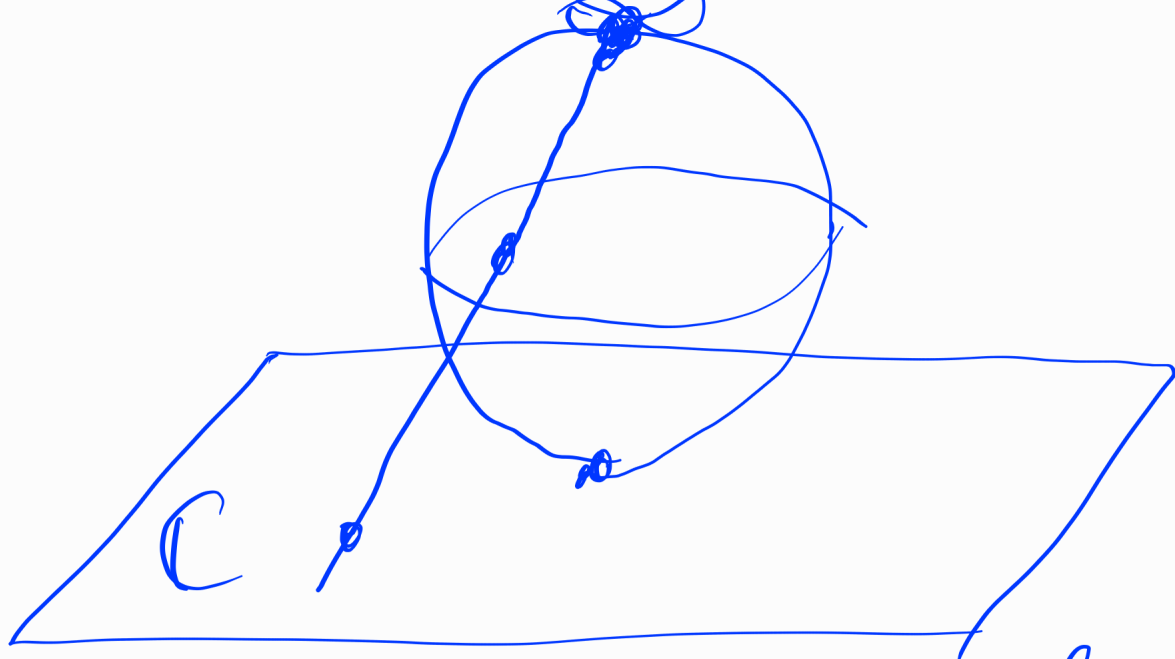
$$x_1 + 2y_1 = 5z_1$$



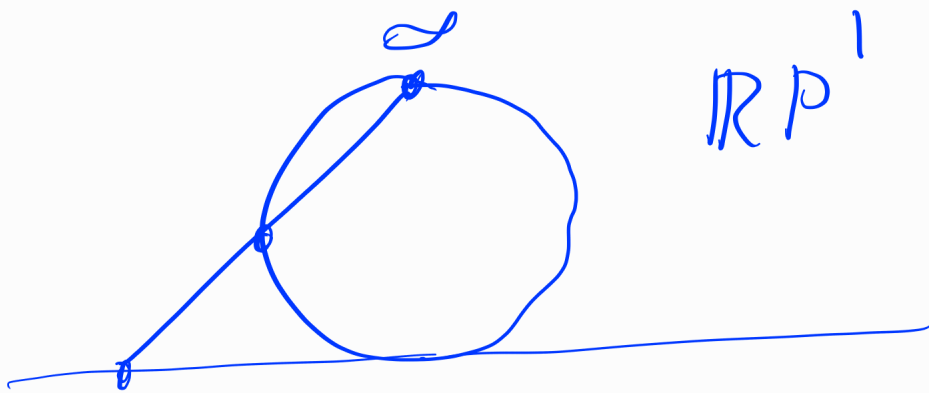
$\mathbb{C}P^n$ same as $\mathbb{R}P^n$,
coordinates are complex.

$\mathbb{C}P^1$ Riemann Sphere

$$\mathbb{C}P^1 = \mathbb{C} \cup \{(1:0)\}$$



$$\mathbb{PGL}_2(\mathbb{C}) \quad x \mapsto \frac{ax+b}{cx+d}$$



$$\mathbb{C} \simeq \mathbb{R}^2$$

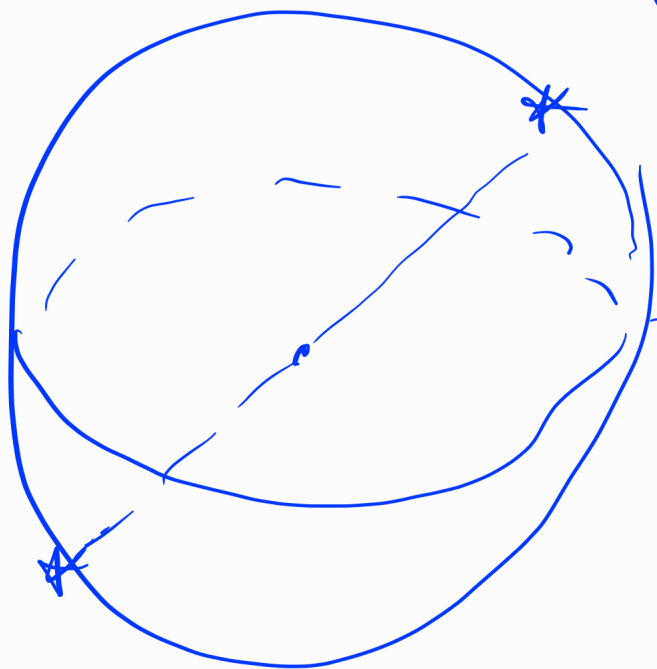
$$\mathbb{CP}^1 \neq \mathbb{RP}^2$$

$$\mathbb{R}P^2 = S^2 / \sim$$

(x, y, z)

z

$(-x, -y, -z)$



$\mathbb{C}P^1, \mathbb{C}P^2$

$\mathbb{C}P^1, \mathbb{C}, \mathbb{C}$ glued
by $\mathbb{C}^*$

$$\mathbb{CP}^1 = (x_1 : x_2) \quad x = x_1/x_2$$

$$\mathbb{C} \cong \{(x : 1)\} = \mathbb{CP}^1 \setminus \{x = \infty\}$$

$$\mathbb{C} \cong \{(1 : t)\} = \mathbb{CP}^1 \setminus \{x = 0\}$$

$$t = \frac{1}{x}, \quad x = \frac{1}{t}$$

$$x, t \in \mathbb{C}^*$$

$$\mathbb{CP}^2 \quad (x_1 : x_2 : x_3)$$

$$x_3 \neq 0 \quad (x : y : 1) \quad U_3$$

$$x_2 \neq 0 \quad (u : 1 : v) \quad U_2$$

$$x_1 \neq 0 \quad (1 : s : t) \quad U_1$$

If x_1, x_2, x_3 are all non-zero:

$$(C^*)^2 \subset U_i \quad i=1,2,3$$
$$(x:y:1) = \left(\frac{x}{y}:1:\frac{1}{y}\right)$$
$$y \neq 0 \quad u = \frac{x}{y}, \quad v = \frac{1}{y}$$

'''

A projective space

$\mathbb{P}^n$ is an example of a toric variety.

"Def." A toric variety is an algebraic variety with an action of a

torus $(\mathbb{C}^*)^n$ that
contains an open Zariski
dense orbit $\simeq (\mathbb{C}^*)^n$.

How does $(\mathbb{C}^*)^n$ act
on $\mathbb{CP}^n$?

$$(x_1 : x_2 : \dots : x_n : x_{n+1}) = x$$

$$(\mathbb{C}^*)^n \ni (\lambda_1, \dots, \lambda_n) = \bar{\lambda}$$

$$\bar{\lambda}(x) = (\lambda_1 x_1 : \dots : \lambda_n x_n : x_{n+1})$$

$$U_{n+1} = "(x_{n+1} = 1)"$$

1/2

$(\mathbb{C}^*)^n$ acting on itself,

This is the orbit of

$$(1:1:\dots:1)$$

$$\bar{\lambda}(1:1:\dots:1) = (\lambda_1:\dots:\lambda_n:1)$$

Ex. Elliptic curve

$$y^2 = x^3 - x \quad x, y \in \mathbb{C}$$

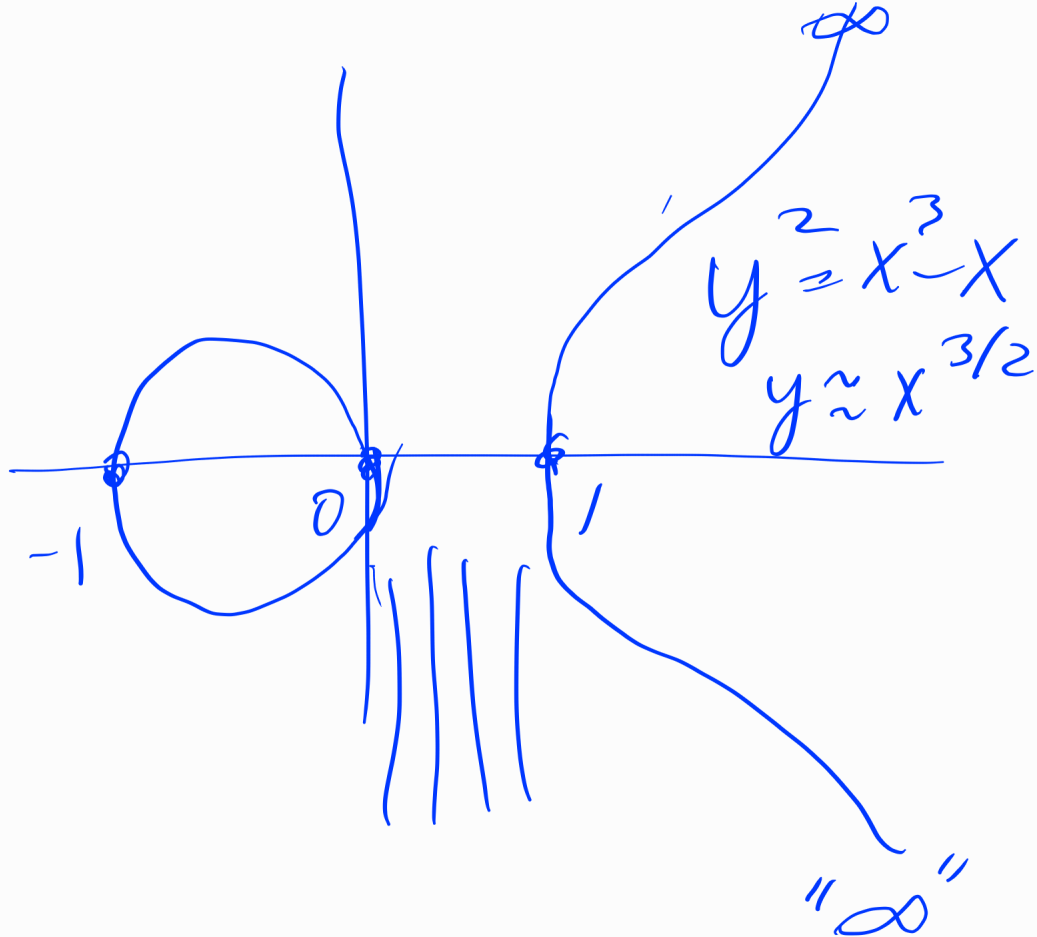
Projectivize it

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3} \quad (x_1:x_2:x_3)$$

$$\frac{x_2^2}{x_3^2} = \frac{x_1^3}{x_3^3} - \frac{x_1}{x_3}$$

$$x_2^2 x_3 = x_1^3 - x_1 x_3^2$$

This makes sense
for $(x_1:x_2:x_3)$



$$x_2^2 x_3 = x_1^3 - x_1 x_3^2 \quad (0:1:0)$$

$$x_3 = 0 \rightarrow x_1 = 0$$

$$(0:1:0)$$

$$u = \frac{x_1}{x_2} \quad v = \frac{x_3}{x_2}$$

$$(u:1:v)$$

$$x_2^2 x_3 = x_1^3 - x_1 x_3^2 \quad \bigg/ x_2^3$$

$$\frac{x_3}{x_2} = \left(\frac{x_1}{x_2} \right)^3 - \left(\frac{x_1}{x_2} \right) \cdot \left(\frac{x_3}{x_2} \right)^2$$

$$v = u^3 - uv^2$$

$v = 0$ line at ∞