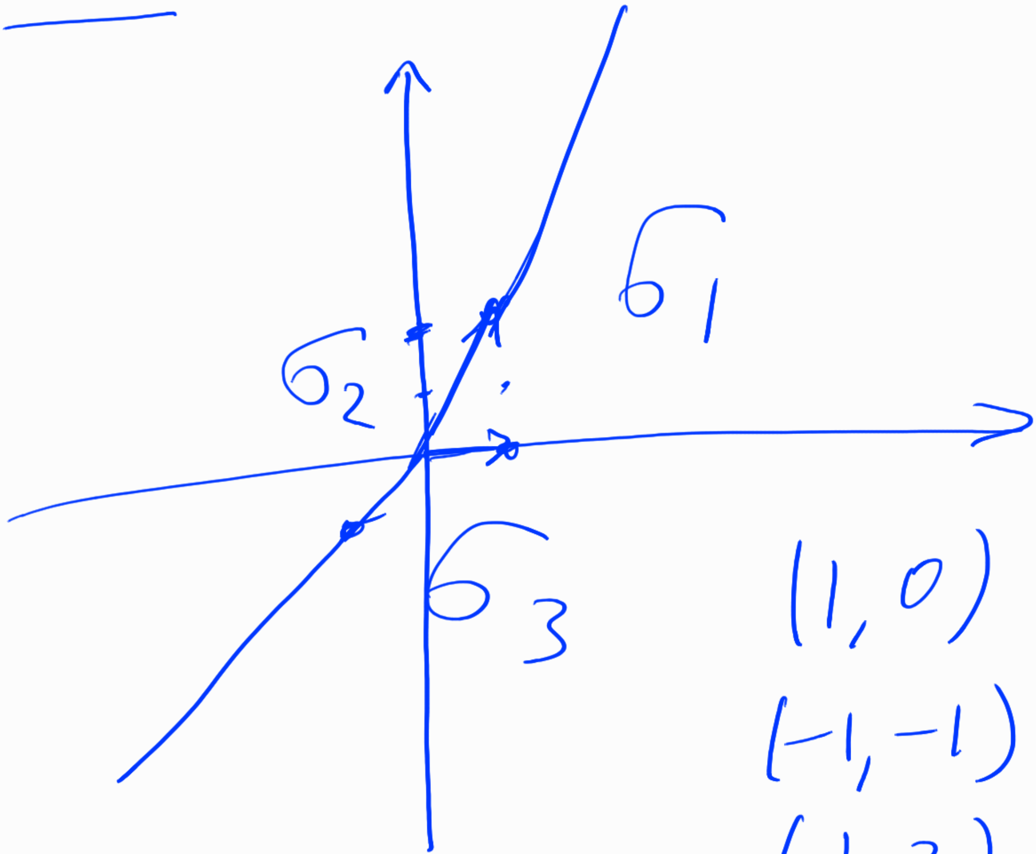
Ex. $d=2$



$(1,0)$
$(-1,-1)$
$(1,2)$

Divisor class groups on toric varieties

$\mathbb{C}[x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$
is UFD.

Every divisor on $(\mathbb{C}^*)^n$ is a divisor of a function

For every D divisor on X

$D \sim \sum a_i E_i$, where

E_j correspond to the

rays in Σ, i.e.

the 1-dimensional cones in Σ.

Given a ray R_i (e.g. generated

by $(1,2)$) the corresponding

affine toric variety

is $m\operatorname{Spec}(\hat{R}_i \cap M)$

Because $\dim R_i = 1$

$\hat{R}_i$ is a half-space

(e.g. $\{(m_1, m_2) \mid 1 \cdot m_1 + 2m_2 \geq 0\}$

The semigroup $\hat{R}_i \cap M$ is generated by:

an integer basis of the co-span$(\hat{R}_i)$ $\left(\{(m_1, m_2) \mid m_1 + 2m_2 = 0\}\right)$

and one more vector "at the distance 1" from the co-span

(e.g. we can take $(m_1, m_2) = (3, -1)$ or anything else s.t. $m_1 + 2m_2 = 1$

As an algebraic variety we get $(\mathbb{C}^*)^{d-1} \times \mathbb{C}$

It contains $(\mathbb{C}^*)^{d-1} \times \mathbb{C}^* = (\mathbb{C}^*)^d$

$(\mathbb{C}^*)^d$ E_i

$$X_{R_i} \setminus \underset{(\mathbb{C}^*)^d}{\underset{\parallel}{X_{\{0\}}}}$$

Given D divisor on X

$D \cap (\mathbb{C}^*)^d$ is a divisor of some f on $(\mathbb{C}^*)^d$

$$D - \mathrm{div}(f) = \sum a_i E_i$$

on X

We can say that the group of divisors on X modulo linear equivalence is the group of equivariant divisors modulo linear equivalence $(\langle E_i \rangle / \sim)$

Lemma.

Suppose f is a rational function on X s.t.

$\mathrm{div}(f) \cap (\mathbb{C}^*)^d = 0$

Then $\underline{f = c \cdot x_1^{m_1} \cdot \ldots \cdot x_d^{m_d}}$

<u>Proof.</u> divisor of f

on $(\mathbb{C}^*)^d$ is 0.

So $f = \frac{f_1}{f_2}$ f_1, f_2 Laurent polynomials

Because it is UFD,

$f =$ unit in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$

So $f = c \cdot x_1^{m_1} \cdot \ldots \cdot x_d^{m_d}$ $\square$

~~Lemma~~

For $f = c\, x_1^{m_1} \cdot \ldots \cdot x_d^{m_d}$

the coeff of $\operatorname{div}(f)$

at $R_i = \langle (n_1, \ldots, n_d \rangle)$

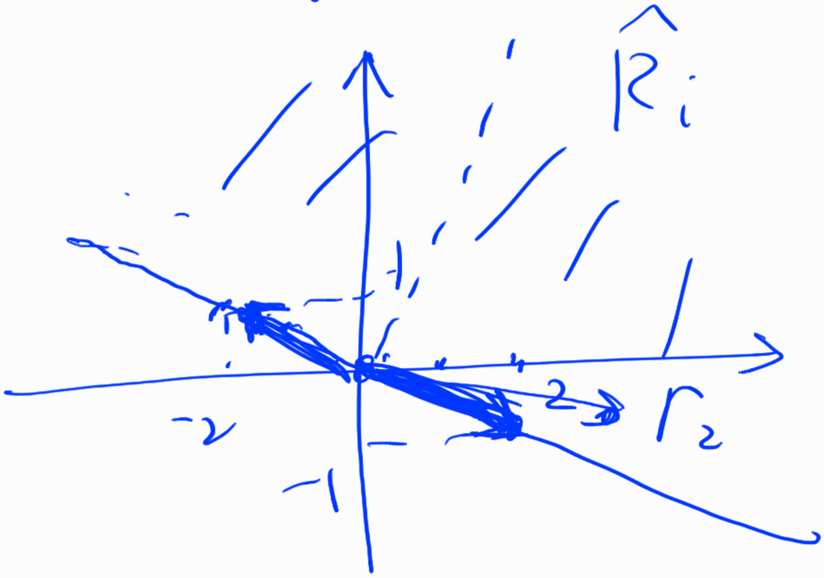
is $n_1 m_1 + \ldots + n_d m_d$.

<u>Proof.</u> $X_{R_i} = \operatorname{mspec}(\mathbb{C}[\hat{R}_i \cap M])$

Ring is isomorphic to

$$\mathbb{C}[y_1^{\pm 1}, \dots, y_{d-1}^{\pm 1}; y_d]$$

<u>Ex.</u> $R_i = \langle (1,2) \rangle$



$m_1 + 2m_2 \geq 0$

$d = 2$

$y_1 = x_1^2 x_2^{-1}$

$(3, -1)$

$$x_1^{m_1} \cdot \dots \cdot x_d^{m_d} = (y_d)^{n_1 m_1 + \dots + n_d m_d} \cdot z$$

where $z = x_1^{i_1} \cdot \dots \cdot x_d^{i_d}$

s.t. $n_1 i_1 + \dots + n_d i_d = 0$.

$(i_1, \dots, i_d)$ is in cospan of $\hat{R}_i$.

Algebraically, if we

localize the ring of $X_{R_i}\left(\mathbb{C}[\hat{R_i} \cap M]\right)$ by the monomials $\mathcal{R}$ the cospan, we get a discrete valuation ring with the uniformizer y_d

---

A monomial $x_1^{m_1} \dots x_d^{m_d}$ corresponds to a linear function

$$N \to \mathbb{Z}$$

$$(n_1 \dots n_d) \mapsto n_1 m_1 + \dots + n_d m_d$$

<u>Theorem.</u> Divisor class group of a toric variety X is a quotient of a free abelian group generated

by (closest to 0 points) on rays R_i of Σ modulo linear functions

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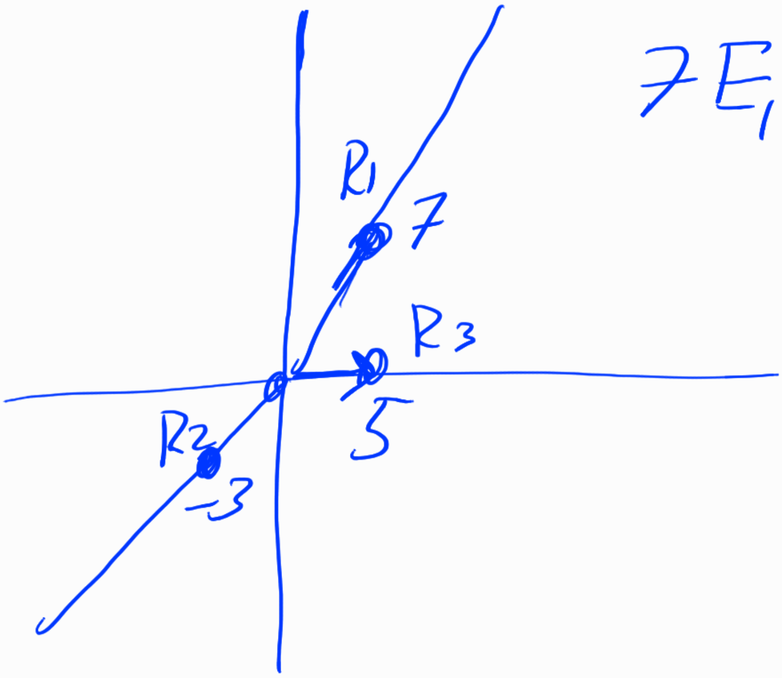
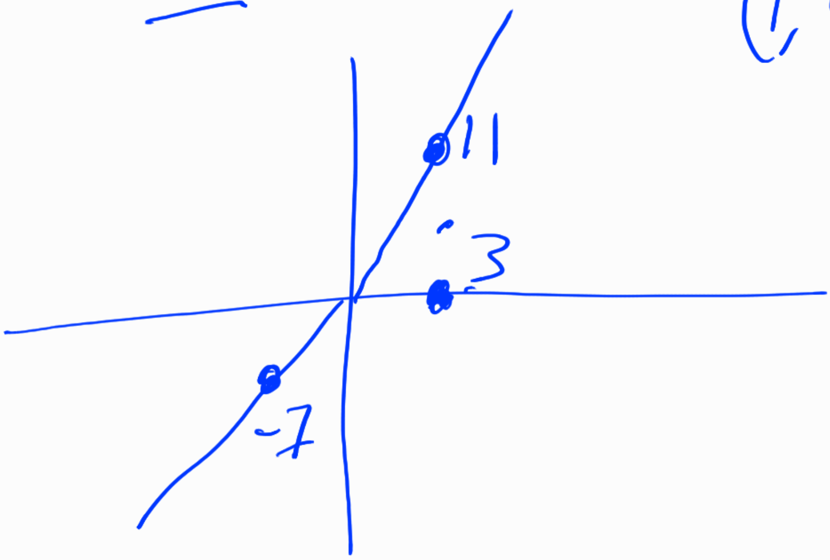


$7E_1 - 3E_2 + 5E_3$

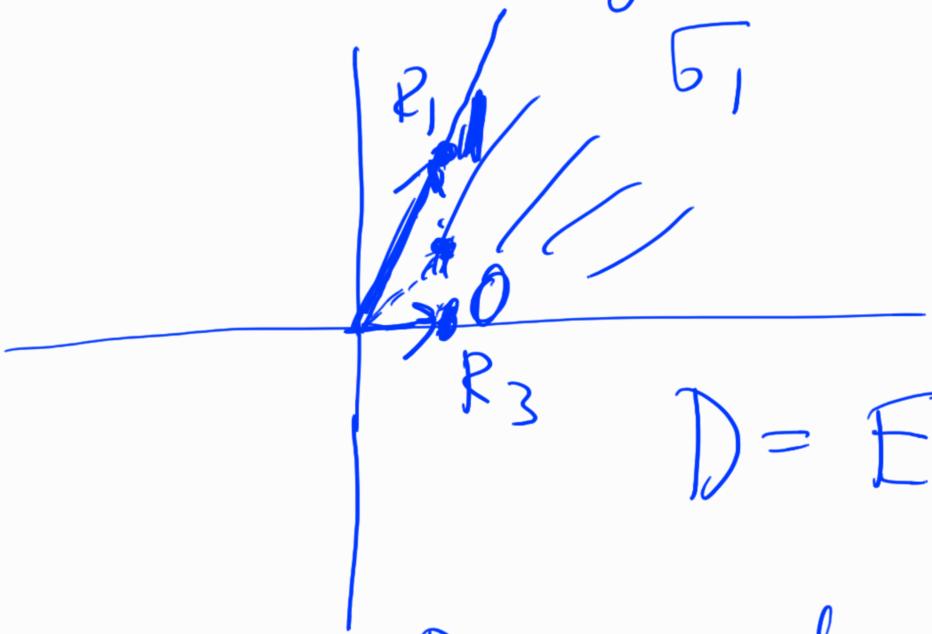
<u>Ex.</u> $\underline{m_1} = 3$, $\underline{m_2} = 4$ $\quad x_1^3 x_2^4$

$(1,0)\cdot(3,4) = 3$



Everything just said works over a not

necessarily complete toric variety.



$D = E_1$

Claim. D is not a divisor of a function on X_{σ_1} but $2D$ is

If $D = \mathrm{div}(x_1^{m_1} x_2^{m_2})$

$$\begin{cases} m_1 + 0\,m_2 = 0 \\ m_1 + 2m_2 = 1 \end{cases} \qquad m_2 = \frac{1}{2}, \quad m_1 = 0$$

$2D = \mathrm{div}(x_2)$

Cartier divisor class group on X:

(cone-wise linear functions on Σ taking integer values on N)

/ linear functions on N

$$\left(\begin{array}{c} \| ? \\ M \end{array}\right)$$