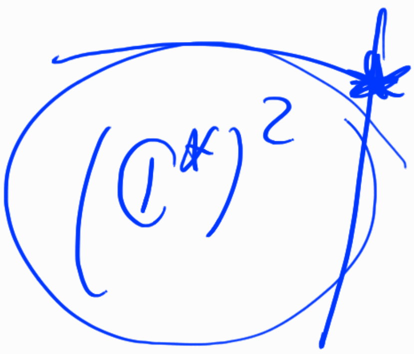
Def Weyl divisor on X is a formal finite sum of irreducible codim 1 subvarieties of X.

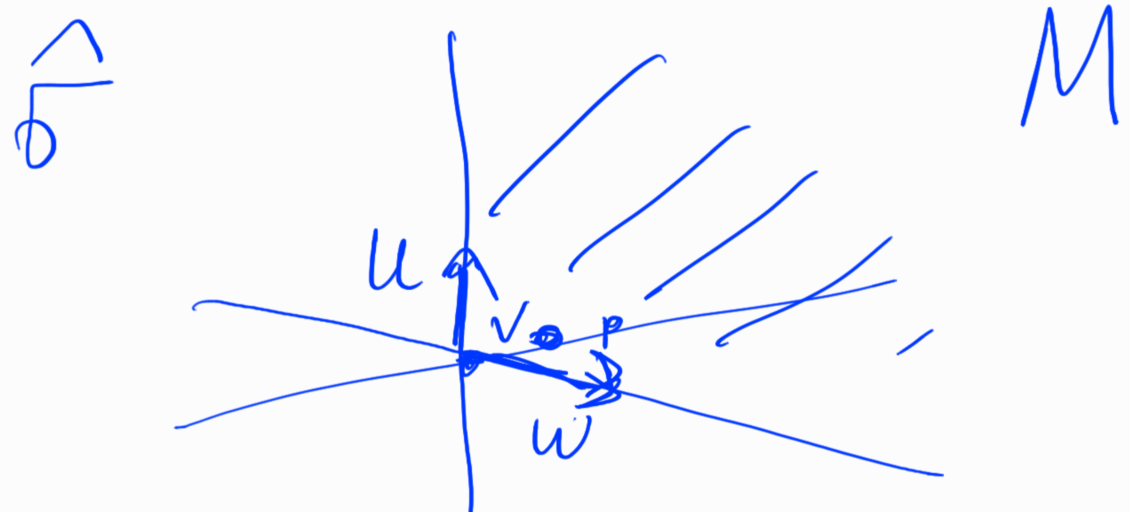
Def. ~~of~~ $f \in K(X)$ (rational function on X) Then $\mathrm{div}(f)$ is the divisor of zeroes and poles on X of f.

Div. Class group (Picard group) ~~of~~ of X is the group of all divisors / group of $\mathrm{div}(f)$

Def. D is a Cartier divisor on X if at every point p of X there is a Zariski neighborhood U of p s.t. on that neighborhood $D = \mathrm{div}(f)$ for some f.

Ex. Toric variety X_σ, $\sigma = \langle (1,0),(1,2) \rangle$, N,



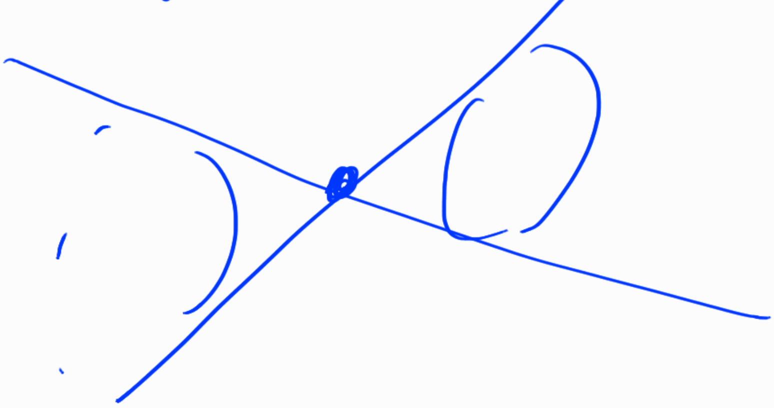


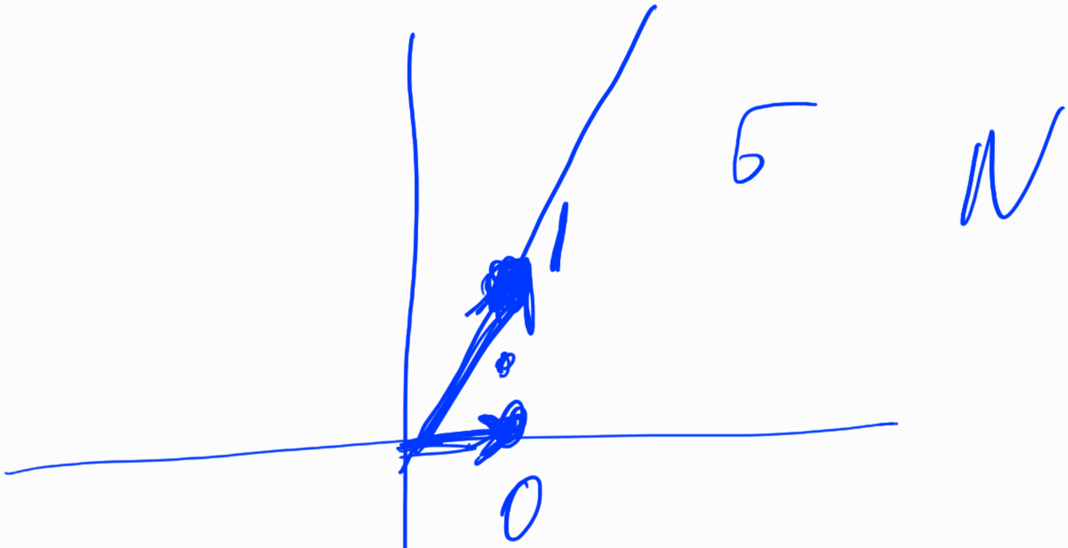
$$\langle (0,1), (2,-1) \rangle$$

$$\begin{cases} u = x_2 \\ v = x_1 \\ w = x_1^2 x_2^{-1} \end{cases} \Rightarrow v^2 = uw$$

$$X = m\,\mathrm{Spec}\left(\mathbb{C}[u,v,w]/(v^2 - uw)\right)$$

$$v^2 = uw$$





Weil divisors/$\sim$

$\cong$

functions on the closest points of rays

/ linear integer-valued functions

If $f = x_1^{m_1} x_2^{m_2}$ s.t.

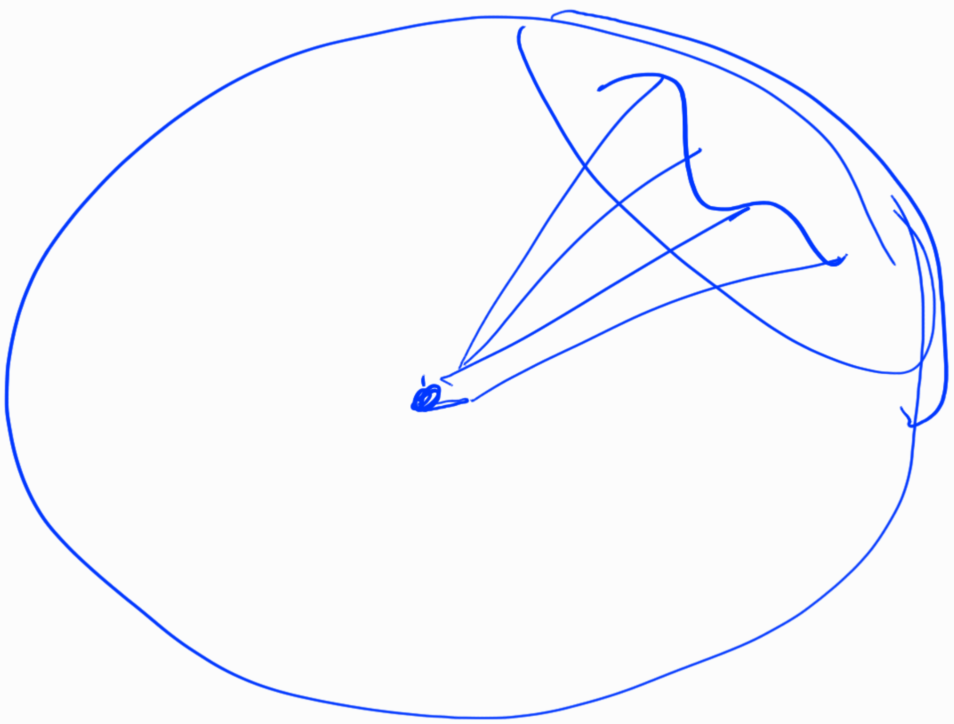
$$\begin{cases} m_1 \cdot 1 + m_2 \cdot 0 = 0 \\ m_1 \cdot 1 + m_2 \cdot 2 = 1 \end{cases}$$

then $\begin{cases} m_1 = 0 \\ m_2 = \frac{1}{2} \end{cases} \notin \mathbb{Z}$

Ex. Cone over a curve

$\mathbb{CP}^3 \supset \mathbb{CP}^2$ at infinity

Suppose we have a curve in that $\mathbb{CP}^2$.



For example

$(X_1 : X_2 : X_3 : X_4)$

$(0 : 0 : 0 : 1)$

$\mathbb{C}P^2 \quad (X_1 : X_2 : X_3 : 0)$

$$X_1^2 X_3 = X_2^3 + X_2 X_3^2$$

On $\mathbb{C}P^3$ this is an equation of a surface, that has a singularity at $(0 : 0 : 0 : 1)$

In A^3 with coordinates

$$x = \frac{X_1}{X_4},\ y = \frac{X_2}{X_4},\ z = \frac{X_3}{X_4}$$

$$X^2 Z = Y^3 + Y Z^2$$

(Weil divisors)/(Cartier div.)

$\cong \mathrm{Pic}(E)$, where

E is the elliptic curves on $(X_1 : X_2 : X_3)$ given by $X_1^2 X_3 = X_2^3 + X_2 X_3^2$

Definitions

X normal irreducible algebraic variety

D is a Weil divisor on X.

Def. D is $\mathbb{Q}$-Cartier if some non-zero

multiple nD is Cartier
$\neq 0$

The smallest positive n s.t. nD is Cartier is the index of D.

<u>Def.</u> X is called $\mathbb{Q}$-factorial if every Weil divisor on X is $\mathbb{Q}$-Cartier

<u>Def.</u> X is Gorenstein if K on X is Cartier

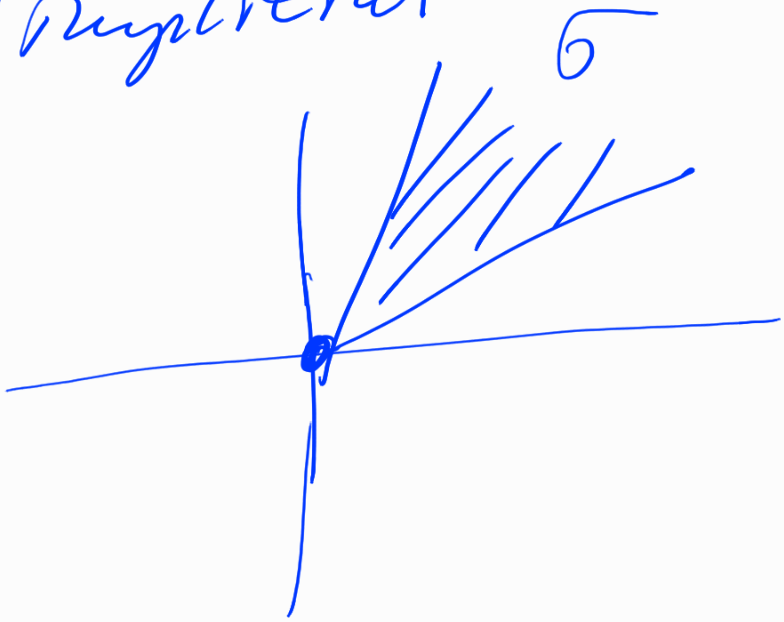
<u>Def.</u> X is $\mathbb{Q}$-Gorenstein if K is $\mathbb{Q}$-Cartier

For toric varieties

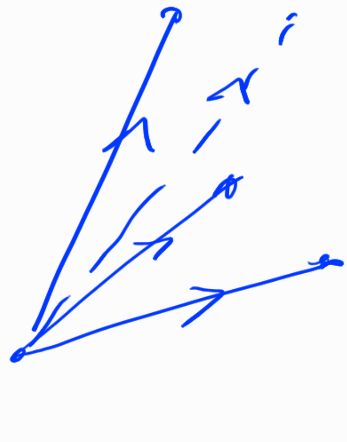
Thm. Suppose X is a toric variety that corresponds to a fan Σ. Then X is smooth iff all cones σ in Σ are 1) simplicial
2) the closest to 0 points on the rays can be augmented to a basis of N.

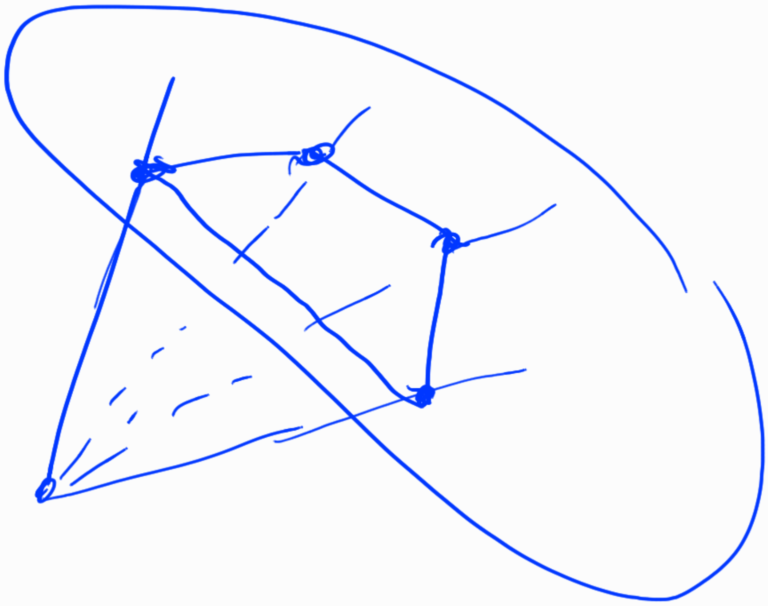
Here simplicial means that if σ has dim m, it is generated by m rays.

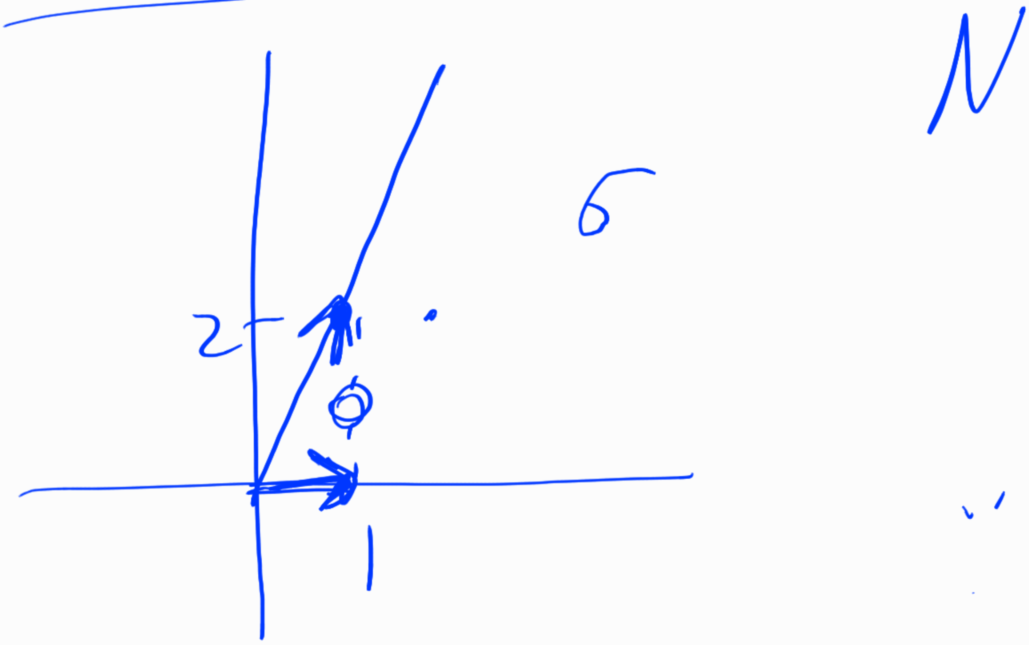
$d=2$ all cones are simplicial

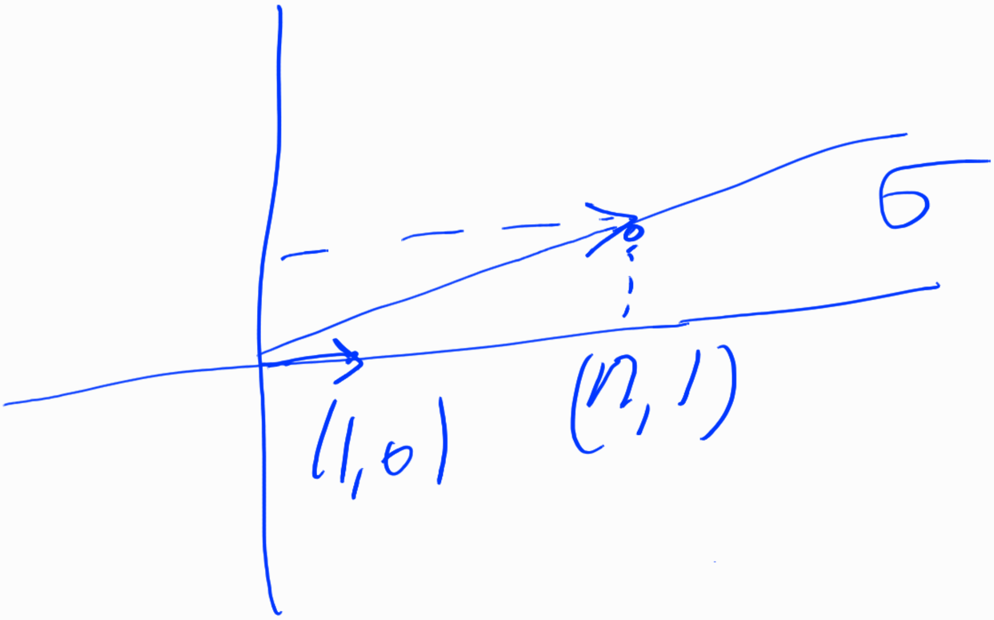


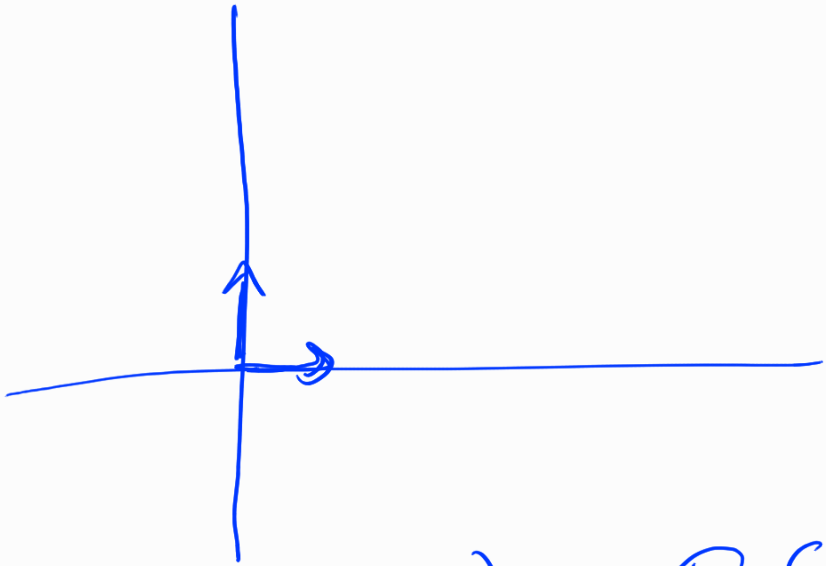
$\underline{d=3}$






N
σ
2
1


σ
(1,0)
(n,1)



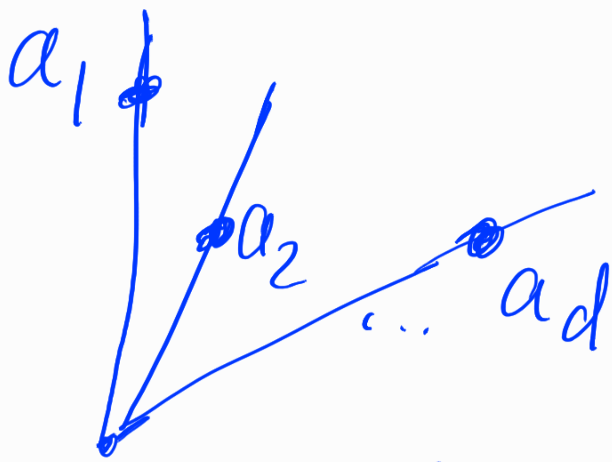
$$\mathbb{C}(\hat{\sigma} \cap M) = \mathbb{C}[y_1, y_2]$$

If the dim of σ is $m \leq d$ and σ satisfies both conditions,

$$m\,\mathrm{Spec}(\mathbb{C}(\hat{\sigma} \cap M)) \cong \mathbb{C}^m \times (\mathbb{C}^*)^{d-m}$$

Thm X_Σ is $\mathbb{Q}$-factorial iff <u>all</u> cones σ in Σ are simplicial

$d=3$



We can find a rational linear function on N s.t. it takes these values on the closest points of the rays. Some multiple of it is an integer-valued linear function on N:

$(m_1, \dots, m_d) \in M$

$(n_1, \dots, n_d) \mapsto n_1 m_1 + \dots + n_d m_d$

$\operatorname{div}(x_1^{m_1} \cdot \dots \cdot x_d^{m_d})$

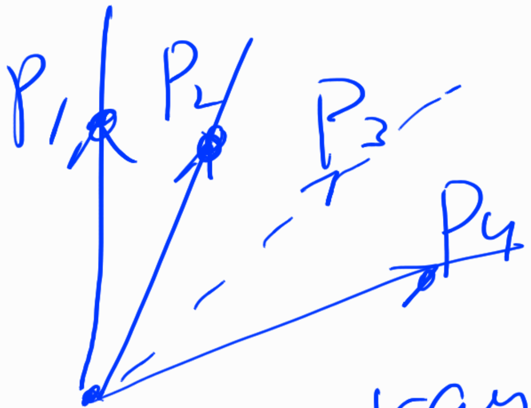
this will be a multiple

of our divisor

$a_1 E_1 + \dots + a_d E_d$

If σ is <u>not</u> simplicial

$N \cong \mathbb{Z}^3$



Suppose rays are P_1, P_2, P_3, P_4

$N = \mathbb{Z}^3$

There a linear relation: $\exists\, c_1, c_2, c_3, c_4 \in \mathbb{Q}$

s.t. $c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P_4 = 0$ in N

where c_1, c_2, c_3, c_4 are not all zero.

We can choose a_1, a_2, a_3, a_4

s.t. $c_1 a_1 + c_2 a_2 + c_3 a_3 + c_4 a_4 \neq 0$

We cannot have a linear function on N with rational coefficients s.t. $f(P_i) = a_i \quad \forall i = 1, 2, 3, 4$

$D = a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4$

Take $c_i \neq 0$

Then E_i is Weil, not $\mathbb{Q}$-Cartier

$P_i \in \mathrm{span}(P_1, \dots P_{i-1}, P_{i+1}, \dots P_d)$

If $f(P_j) = 0 \ \forall j \neq i$,

$$f(P_i) = 0.$$

<u>Thm.</u> The canonical divisor class K on X_Σ is $-\sum_i E_i$ (sum over all rays).

<u>Proof.</u>

$$\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \dots \wedge \frac{dx_d}{x_d}$$

<u>Ex.</u> $\mathbb{C}P^1$ $\quad \dfrac{dx}{x} = -\dfrac{d(1/x)}{(1/x)}$

pole of order 1 at $x = 0$

pole of order 1 at $x = \infty$

$x_1 \ldots x_d$

$y_1 \ldots y_d$ monomials, generating the same group by multiplication

$$\frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_d}{y_d} = \pm\, \omega$$