$X = X_\Sigma$

ray in Σ

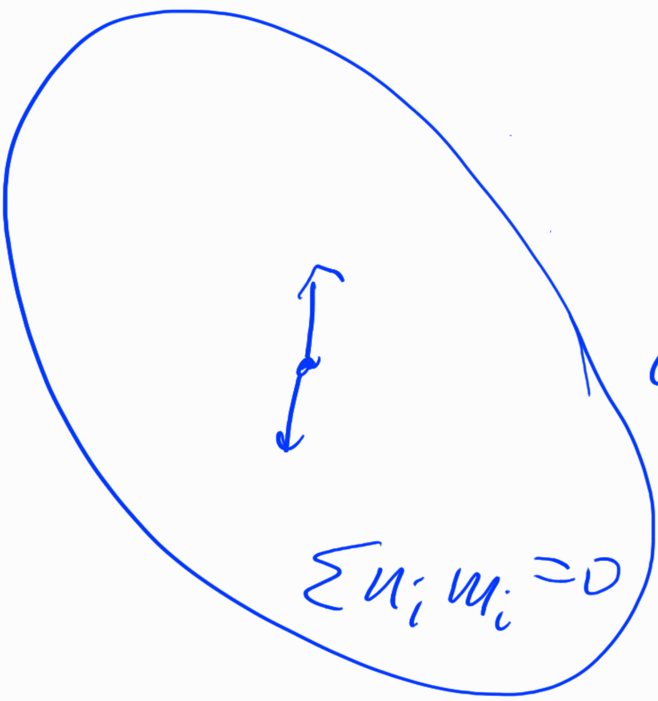
$R_i = \langle n_1, n_2, \ldots, n_d \rangle$ $\quad N$

$\gcd(n_1, n_2, \ldots, n_d) = 1$

$\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \ldots \wedge \frac{dx_d}{x_d}$

$E_i = X_{R_i} \setminus (\mathbb{C}^*)^d$

$\hat{R}_i = \{(m_1, m_2, \ldots, m_d) \mid \sum_{i=1}^{d} n_i m_i \geq 0\}$



$\mathrm{cospan}(\hat{R}_i)$

$\shortparallel$

$\overbrace{\mathrm{span}(R_i)}$

$M \cap \mathrm{Cospan}(\hat{R}_i)$ is generated

as a group by

$y_1, \dots, y_{d-1}$

as a semigroup:

$y_1, y_2, \dots, y_{d-1}, y_1^{-1}, \dots, y_{d-1}^{-1}$

Take $y_d = (m_1, \dots, m_d)$

s.t. $n_1 m_1 + \dots + n_d m_d = 1$

$\hat{R}_i \cap M = \langle y_1^{\pm 1}, \dots, y_{d-1}^{\pm 1}; y_d \rangle$

$M = \langle y_1^{\pm 1}, \dots, y_{d-1}^{\pm 1}; y_d^{\pm 1} \rangle$

~~At~~ a generic point

of E_i

$y_1, \dots, y_d$ are the

local coordinates

and $E_i = \{y_d = 0\}$

$$y_k = \prod_{j=1}^{d} x_j^{a_{jk}} \qquad a_{jk} \in \mathbb{Z}$$

$$\det(a_{jk}) = \pm 1$$

<u>Lemma.</u> $\frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_d}{x_d}$

$$= (\pm 1) \cdot \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_d}{y_d}$$

$$\begin{pmatrix} \log y_1 \\ \vdots \\ \log y_d \end{pmatrix} = (a_{jk}) \begin{pmatrix} \log x_1 \\ \vdots \\ \log x_d \end{pmatrix}$$

<u>Ex.</u> $d = 2$

$$\begin{cases} y_1 = x_1^2 x_2^1 \\ y_2 = x_1^3 x_2^1 \end{cases} \quad \begin{cases} x_1 = y_1^{-1} y_2 \\ x_2 = y_1^3 y_2^{-2} \end{cases}$$

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$$

$$\begin{cases} \log x_1 = -\log y_1 + \log y_2 \\ \log x_2 = 3\log y_1 - 2\log y_2 \end{cases}$$

$$\begin{cases} \dfrac{d\cancel{x}_1}{x_1} = -\dfrac{dy_1}{y_1} + \dfrac{dy_2}{y_2} \\ \dfrac{dx_2}{x_2} = 3\dfrac{dy_1}{y_1} - 2\dfrac{dy_2}{y_2} \end{cases}$$

$$\begin{bmatrix} \dfrac{dx_1}{x_1} \\ \dfrac{dx_2}{x_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} \dfrac{dy_1}{y_1} \\ \dfrac{dy_2}{y_2} \end{bmatrix}$$

$$\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} = \underset{\substack{\| \\ -1}}{\det\begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}} \left(\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \right)$$

$$\omega = (\pm 1) \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_d}{y_d}$$

$$= \underbrace{\pm \frac{1}{y_1 \cdots y_{d-1}}}_{} \cdot \frac{1}{y_d} dy_1 \wedge \ldots \wedge dy_d$$

unit at a generic point on E_i $\quad E_i = \{y_d = 0\}$

ω has a pole of order 1 at E_i. $\square$

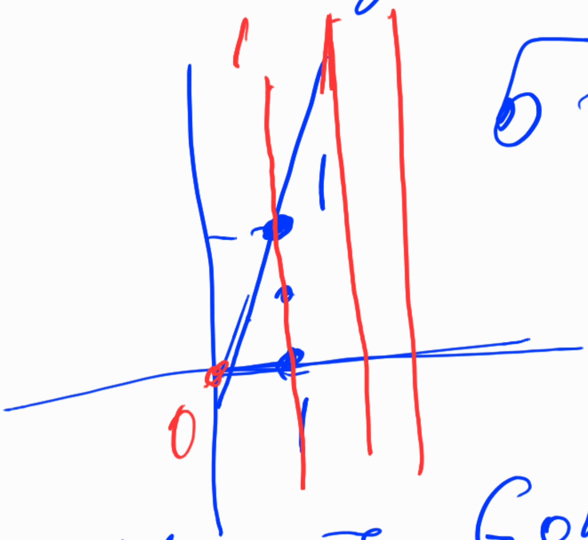
Def. X is Gorenstein if K_X is Cartier

Def. X is $\mathbb{Q}$-Gorenstein if nK_X is Cartier for some $n \in \mathbb{Z}_{>0}$.

Combinatorially:

<u>Thm.</u> X_Σ is Gorenstein iff $\forall$ cone $\sigma \in \Sigma$ the <u>rational linear function</u> on $N \cap \mathrm{span}(\sigma)$ that takes value 1 at the closest points of the rays is integer

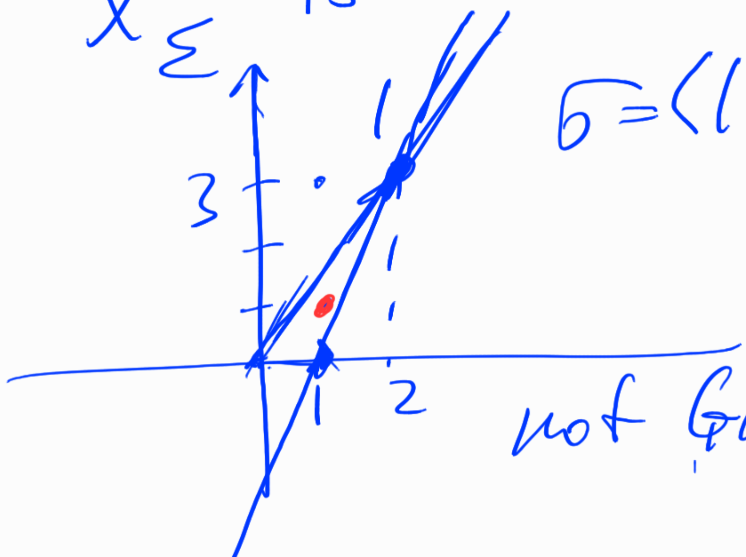
<u>Ex.</u>



$\sigma = \langle (1,0), (1,2) \rangle$

This X_Σ is Gorenstein

<u>Ex.</u>



$\sigma = \langle (1,0), (2,3) \rangle$

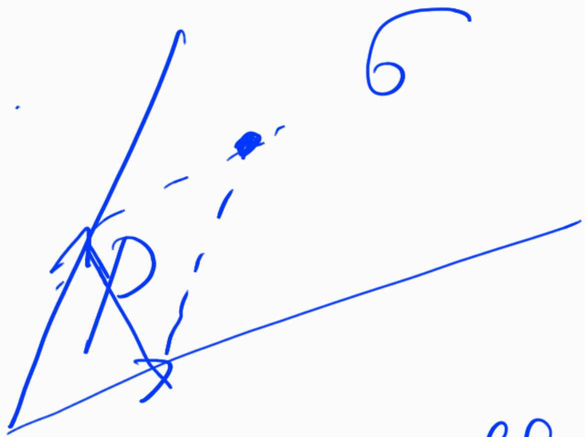
not Gorenstein

The linear function

is $(n_1, n_2) \mapsto m_1 n_1 + m_2 n_2$

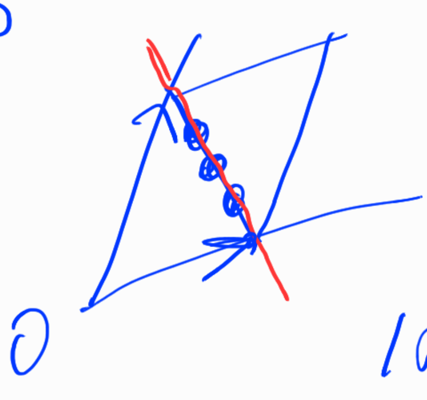
$$\begin{cases} m_1 = 1 \\ 2m_1 + 3m_2 = 1 \end{cases} \qquad \begin{matrix} m_1 = 1 \\ m_2 = -\frac{1}{3} \notin \mathbb{Z} \end{matrix}$$

In dim 2.



X_σ is smooth iff P has no integer points (in N) except for vertices.

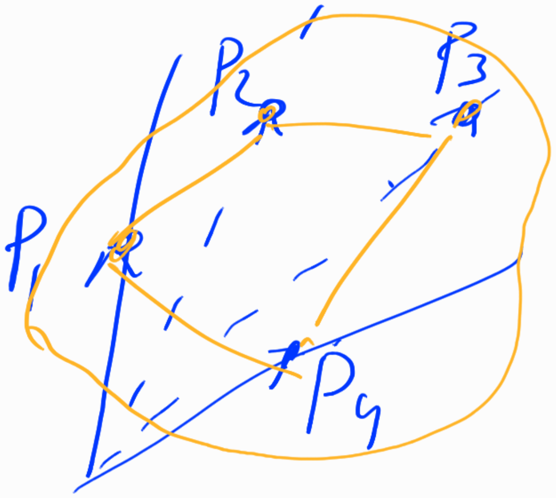
X_σ is Gorenstein iff



all points are vertices or are on this line, connecting rays

What if this rational linear function does not exist?

σ $\dim = 3$

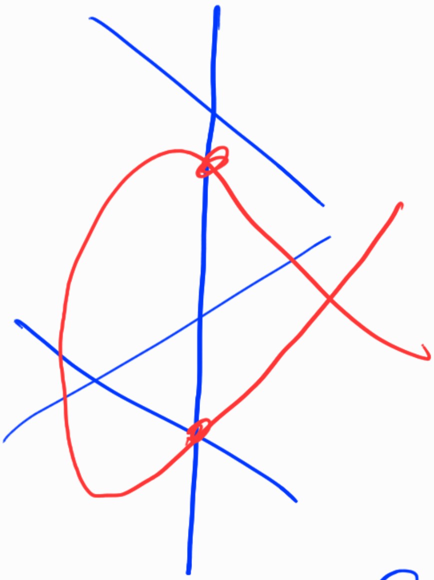


If all P_i are on one hyperplane, then this linear function exists.

In this case X_σ is $\mathbb{Q}$-Gorenstein.

smooth $\subset$ Gorenstein $\subset$ $\mathbb{Q}$-Gorenstein

Blow-up of a point



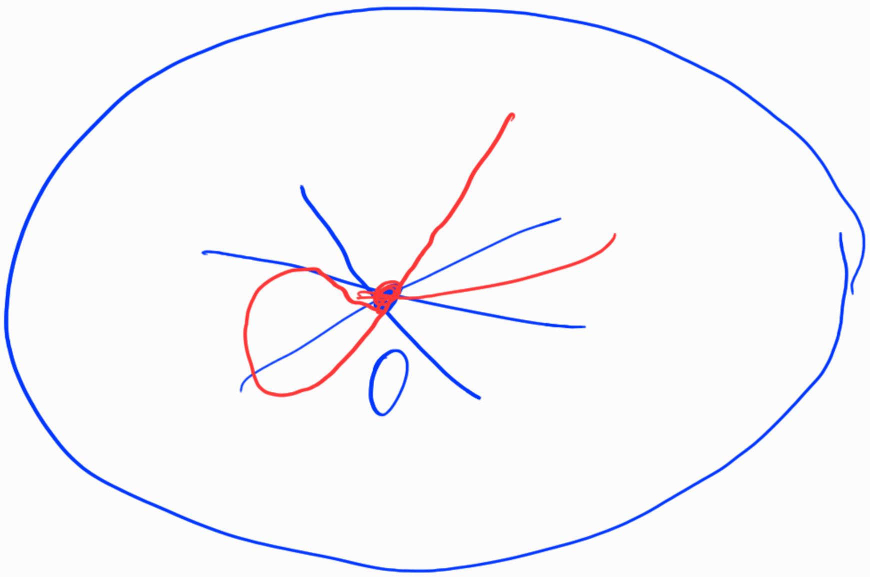
X

$E \cong \mathbb{CP}^1$

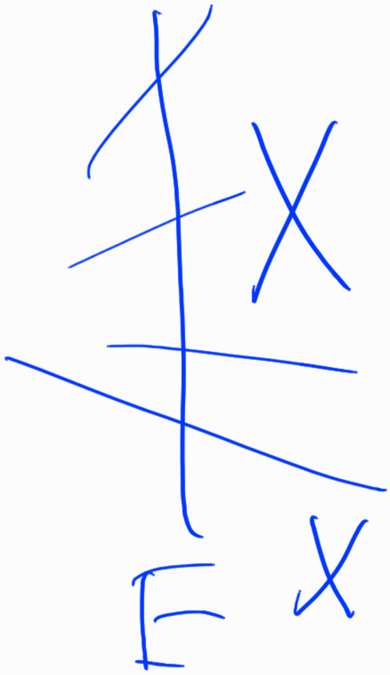
$X \setminus E \overset{\pi}{\cong} \mathbb{C}^2 \setminus \{0\}$

$\pi(E) = 0.$

$\Big\downarrow \pi$



$\mathbb{C}^2$

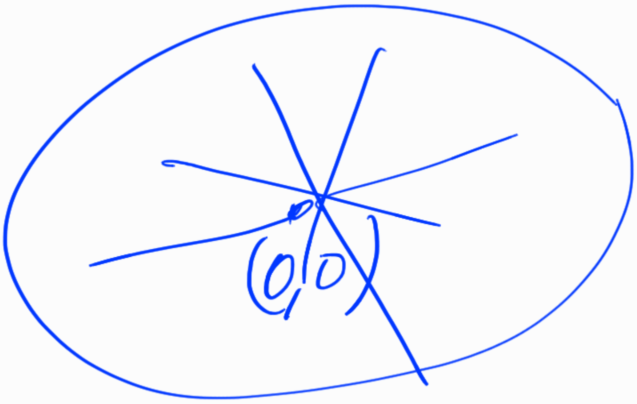


$X \subset \mathbb{C}^2 \times \mathbb{CP}^1$

$((x,y),(s:t))$

$X = \{xt = ys\}$

$\pi \downarrow$ $\qquad\qquad \downarrow$ projection to $\mathbb{C}^2$



$\mathbb{C}^2$

(x,y)

If $(x,y)=(0,0)$

$\forall (s:t)$ $\quad \underset{0}{\overset{}{\underset{\shortparallel}{xt}}} = \underset{0}{\underset{\shortparallel}{ys}}$

If $(x,y)\neq(0,0)$

then $xt = ys \iff (x:y)=(s:t)$ on $\mathbb{CP}^1$

$\forall (s:t) \in \mathbb{C}P^1$

$\{(x,y),(s:t) \mid xt = ys\}$

$(x,y) = \lambda \cdot (s,t)$

$\lambda \in \mathbb{C}$

X is smooth,

X is a union of two

$\mathbb{C}^2$:

$\mathbb{C}P^1 = \{t \neq 0\} \cup \{s \neq 0\}$

$X \cap \{t \neq 0\}$ $\quad z = \frac{s}{t}$

$(x,y),(s:t)$ $\quad xt = ys$

$x = yz$

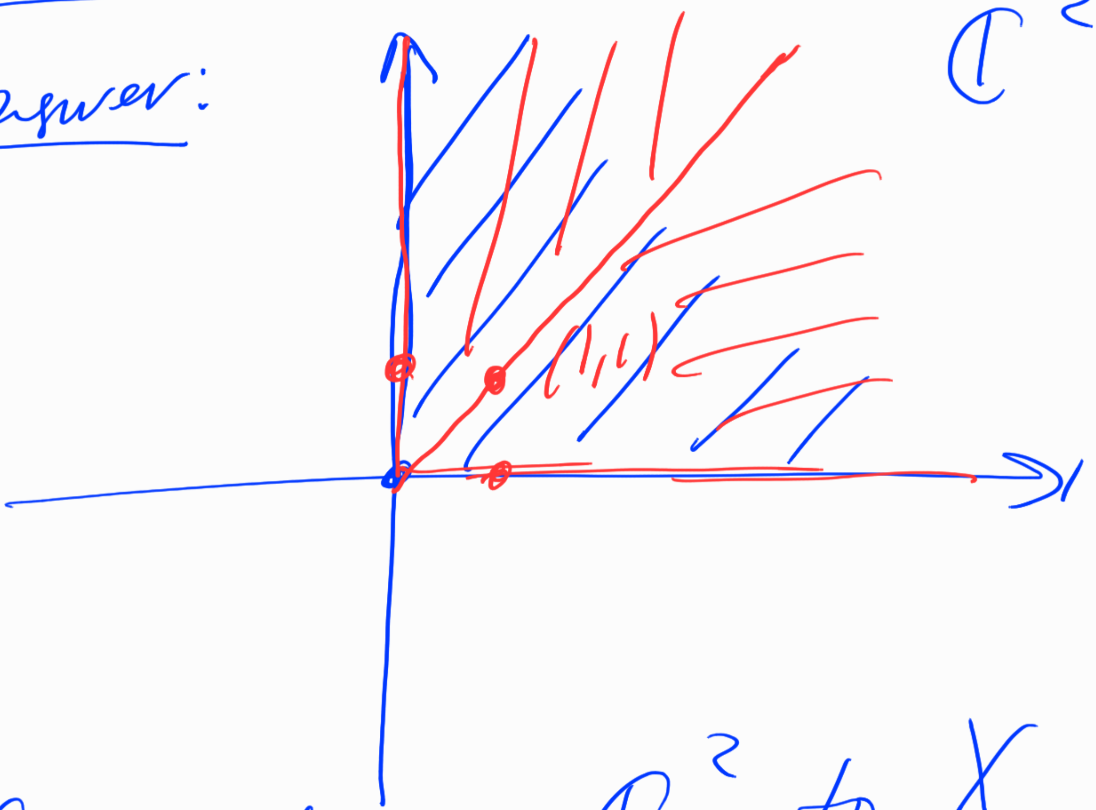
$z = \frac{x}{y}$

$\mathbb{C}^2 \quad \mathbb{C}(x,y)$

$X \quad \mathbb{C}(x,y)$

What is this toroically

Answer:



Going from $\mathbb{C}^2$ to X corresponds to a subdivision of a cone.