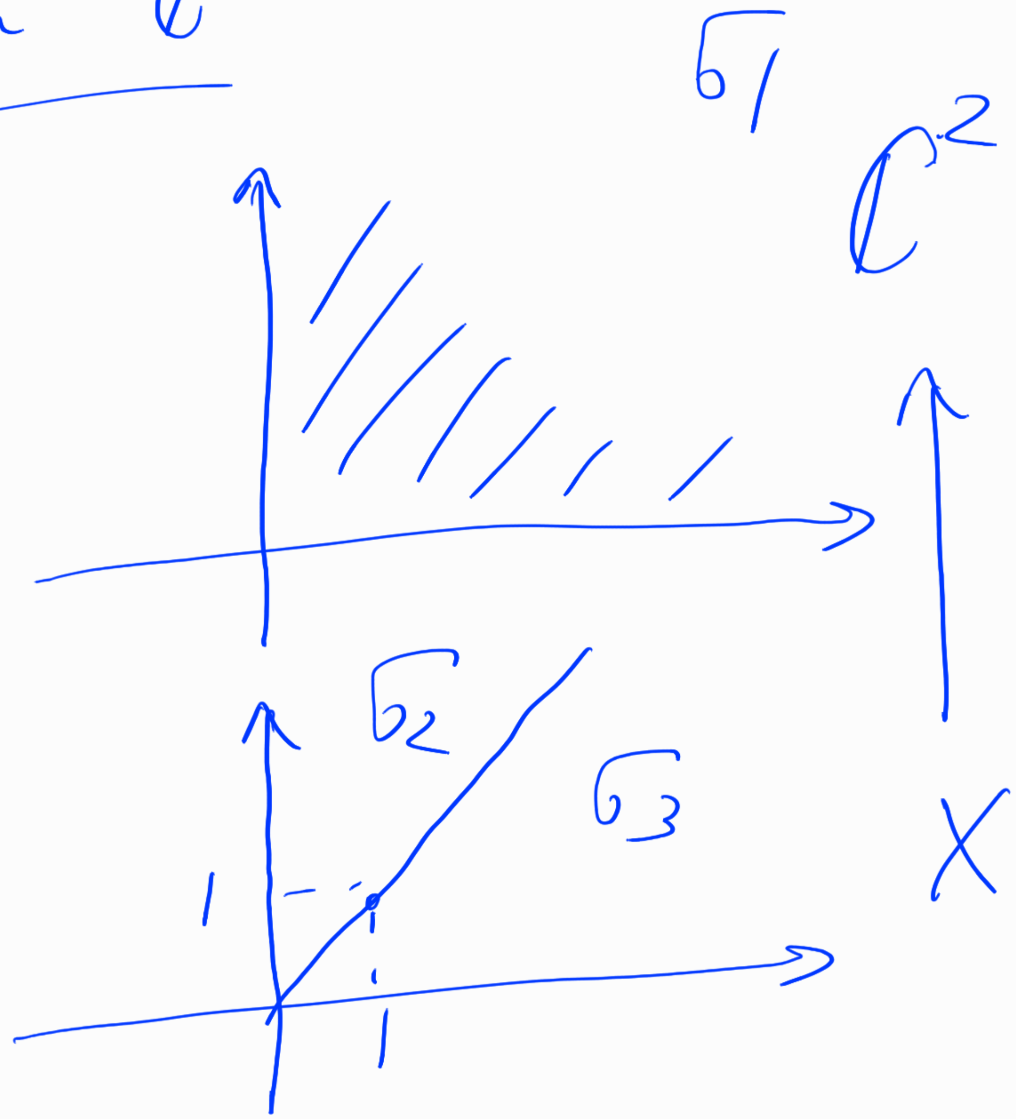
Blow-up of a point in $\mathbb{C}^2$



A map between toric varieties that commutes with the action of the tores can be described by the following:

$$X_1 \xrightarrow{f} X_2$$

$$X_1 = X_{\Sigma_1}, \quad \Sigma_1 \text{ fan in } N_1$$

$$X_2 = X_{\Sigma_2}, \quad \Sigma_2 \text{ fan in } N_2$$

$$K(X_1) \longleftarrow K(X_2)$$

$$M_1 \longleftarrow M_2$$

$$N_1 \longrightarrow N_2$$

Combinatorial data:

$$f : N_1 \longrightarrow N_2$$

s.t. $\forall \sigma_1 \in \Sigma_1 \ \exists \sigma_2 \in \Sigma_2$

s.t. $f(\sigma_1) \subseteq \sigma_2$

$$f(\widehat{\sigma_1}) \supseteq \widehat{\sigma_2}$$

We get a map from $M_2 \cap \hat{\sigma}_2$ to $\hat{\sigma}_1 \cap M_1$

$$\hat{\sigma}_2 \cap M_2 = \{\vec{m} \in M_2 \mid \forall \vec{n}_2 \in \sigma_2 \;\; \vec{m}(\vec{n}_2) \geq 0\}$$

In particular, $\forall \vec{n}_2$ of the form $f(\vec{n}_1)$ we have $\vec{m}(f(\vec{n}_1)) \geq 0$ $\forall m \in \hat{\sigma}_2$

$$\vec{m} \circ \vec{f} \in \hat{\sigma}_1$$

This gives a map from

$$\mathbb{C}[\hat{\sigma}_2 \cap M_2] \longrightarrow \mathbb{C}[\hat{\sigma}_1 \cap M_1]$$

$$X_{\sigma_1} \longrightarrow X_{\sigma_2}$$

Ex. $\mathbb{C} \longrightarrow X_\Sigma$

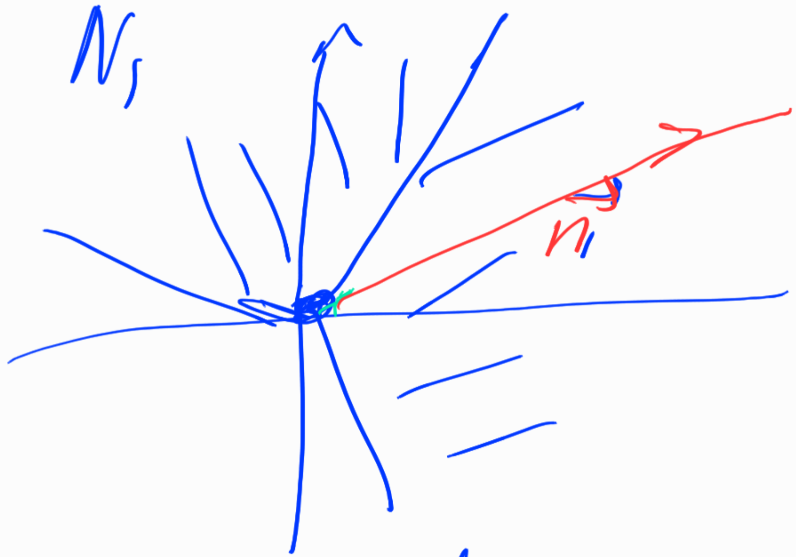
$\mathbb{C}$ $\quad \bullet \xrightarrow{\sigma_1}$ $\quad N_1$

$$\Sigma = \{\sigma_1, \{0\}\}$$

$f(1) \in N$
$\underset{n}{\wedge}$

$f'(1) = \vec{u}$

N_1



<u>Ex.</u> dim $= d$

X_Σ, N

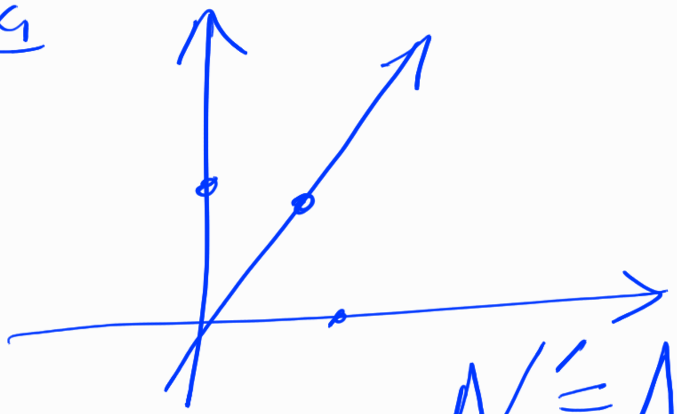
Consider a bigger lattice:

$N' \supseteq N$ $[N' : N] = m$

Then $X_{\Sigma, N} \longrightarrow X_{\Sigma, N'}$

$N \hookrightarrow N'$

Ex.



$N = \mathbb{Z}^d$

$\sigma = \{\forall x_i \geq 0\}$

$N' = N + \frac{1}{r}(a_1, \dots, a_d)$

$$\mathbb{C}^d \longrightarrow X_{\sigma, N'}$$

Cyclic quotient singularity

$\mathbb{C}^d$ coordinates $x_1, \dots, x_d$

$\mu_r = \{\mu \in \mathbb{C}^* \mid \mu^r = 1\}$

μ_r acts on $\mathbb{C}^d$

by $\mu \cdot (x_1, \dots x_d) = (\mu^{a_1} x_1, \dots, \mu^{a_d} x_d)$

$a_1, a_2, \dots, a_d \in \mathbb{Z}$ $(\mathbb{Z}/r\mathbb{Z})$

$X = \operatorname{mspec}(R)$

$G : X \circlearrowleft$

$G : R \circlearrowleft$

$X/G \overset{\text{def}}{=} \text{mSpec}(R^G)$

$\{f \in R \mid \forall g \in G \;\; g \circ (f) = f\}$

Ex. $d = 2 \quad r = 5$

$a_1 = 2, \; a_2 = 1$

Singularity of the type

$\frac{1}{5}(2,1)$

$R = \mathbb{C}[x_1, x_2]$

$\forall \mu \in \mu_5 \quad \mu(x_1^{m_1} x_2^{m_2})$

$= \mu^{2m_1 + m_2} \circ (x_1^{m_1} x_2^{m_2})$

If $f = \sum a_{ij} x_1^i x_2^j$

$\mu(f) = \sum a_{ij} \mu^{2i+j} x_1^i x_2^j$

R^G — ? $\quad \forall a_{ij} \neq 0$

$2i + j \equiv 0 \pmod r$

$\frac{2i}{r} + \frac{j}{r} \in \mathbb{Z}$

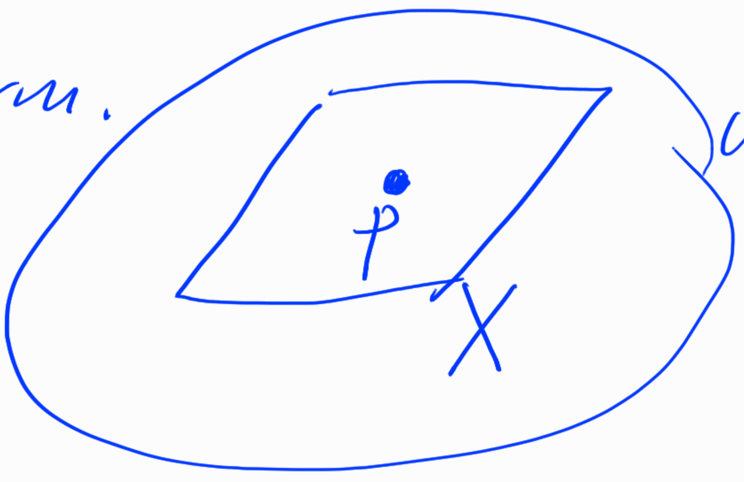
$$N = \mathbb{Z}^2$$

$$N' = \left\langle \mathbb{Z}^2, \tfrac{1}{5}(2,1) \right\rangle$$

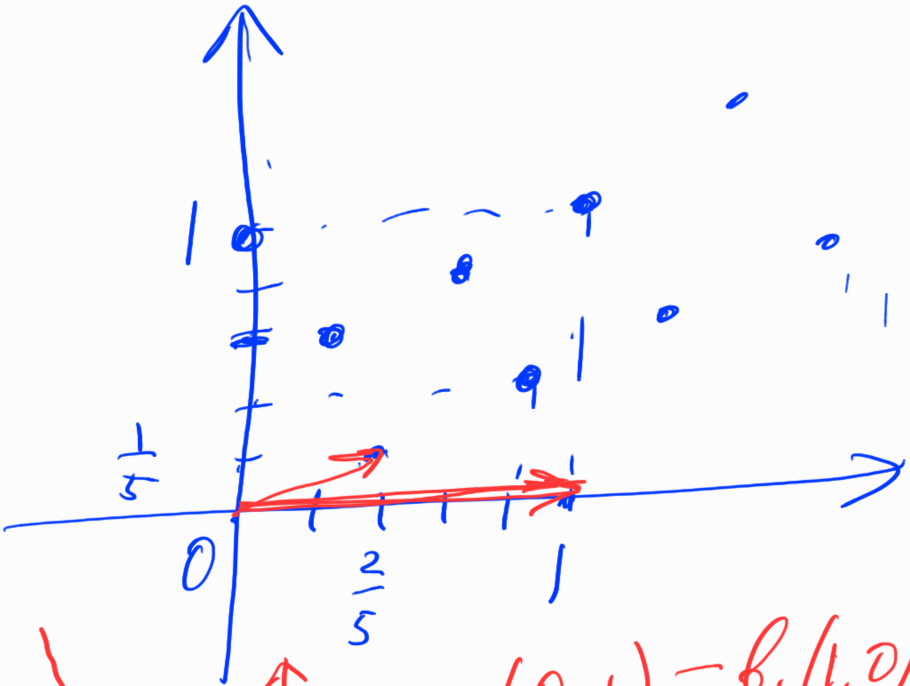
$$M' = \left\{ (m_1, m_2) \in \mathbb{Z}^2 \;\middle|\; (m_1, m_2) \cdot \left(\tfrac{1}{5}(2,1)\right) \in \mathbb{Z} \right\}$$

$$M' = \left\{ \underset{\in \mathbb{Z}^2}{(m_1, m_2)} \;\middle|\; \frac{2m_1}{r} + \frac{m_2}{r} \in \mathbb{Z} \right\}$$

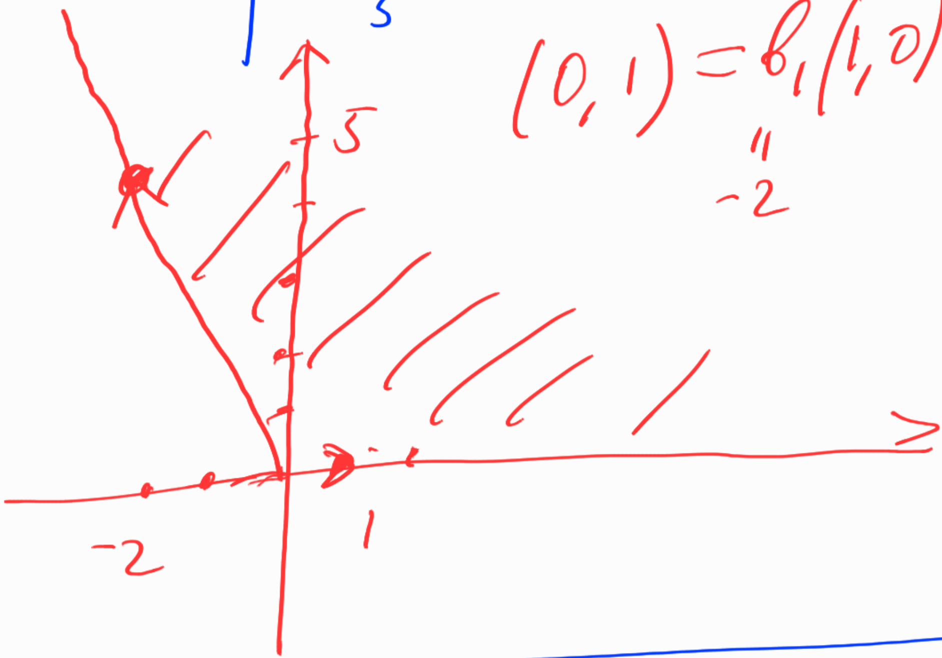
"Locally", any action of μ_r on any X smooth (not toric) is of this form.



$\mu : T_p$



$$(0,1) = \underset{\underset{-2}{\parallel}}{b_1}(1,0) + \underset{\underset{5}{\parallel}}{b_2}\left(\tfrac{2}{5}, \tfrac{1}{5}\right)$$



$$H^0(D) = \{0\} \cup \{f \in K(X) \mid \operatorname{div}(f) + D \geq 0\}$$

$X = X_\Sigma$ $\quad D$ is given by

$\sum a_i E_i$ $\qquad E_i \leftrightarrow R_i$

$$f = \frac{f_1}{f_2}$$

Because D is outside of $(\mathbb{C}^*)^d$, f_2 is monomial

$f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$

$\forall E_i \quad \operatorname{div}(f) = \inf\left\{ \vec{m} \cdot \vec{n}_i \,\middle|\, a_{\vec{m}} \neq 0 \right\}$

$R_i = \langle (n_1 \ldots n_d) \rangle$

$\shortparallel$

$\vec{n}_i$

$\operatorname{div}(f) + D \geq 0$

$\forall i \quad \vec{m} \cdot \vec{n}_i$

$H^0(D)$ is spanned by $\underline{x}^{\vec{m}}$ where $\vec{m}$ is cut out by finitely many linear inequalities.

X complete:
we get compact
convex rational polytope.

D is Cartier: $\Rightarrow$
polytope will have
vertices in M.

P polytope in $\mathbb{Z}^d$
compact, vertices in $\mathbb{Z}^d$

$\#(nP \cap \mathbb{Z}^d)$ polynomial
in n

$n \in \mathbb{Z}$

(Ehrhart polynomial)