Classification Problem

dim 1   genus, X complete, smooth

$\mathbb{CP}^1$ $g=0$

$g=1$ elliptic

$g\geq 2$ "general case"

K: $\deg(K)=2g-2$

$g=0$ $\deg(K)=-2$ $-K$ ample

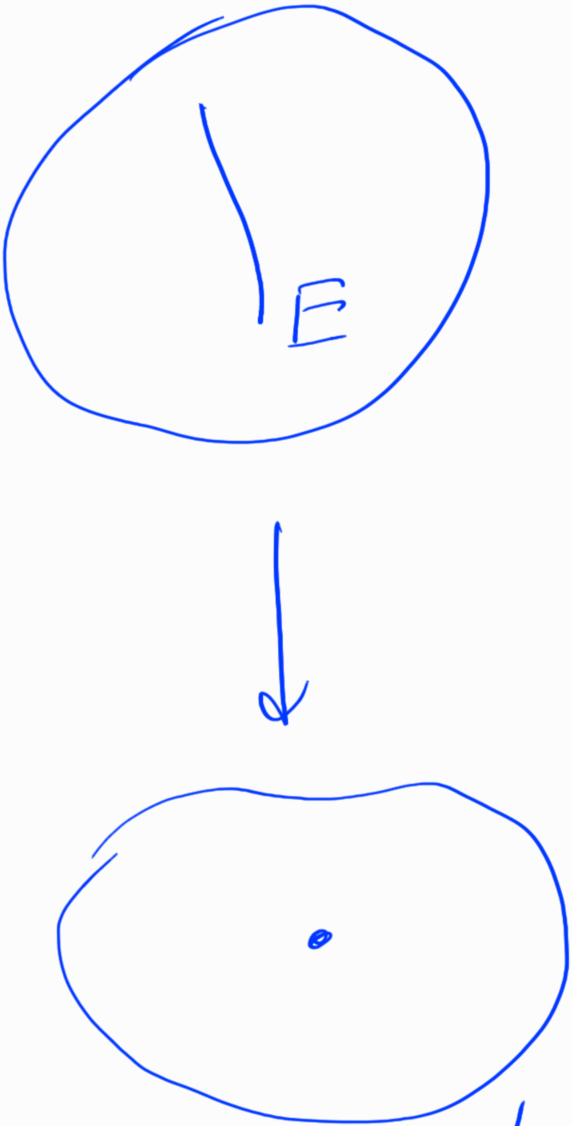
$g=1$ $K=0$

$g\geq 2$ K ample

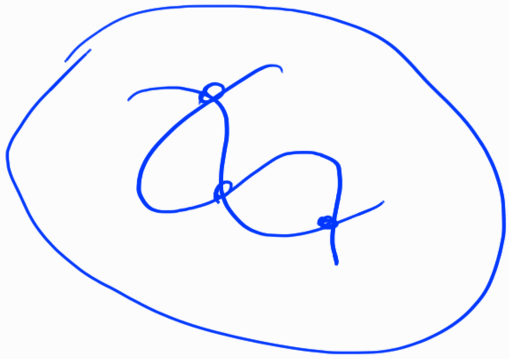
growth of $h^0(nK)$

dim 2

Blow-up / blow-down phenomenon



Caselnuovo criterion



$E \cong \mathbb{CP}^1$, $E^2 = -1$

known as (-1)-curves

$E \cdot K = -1$

$$2g_a(E)-2 = E\cdot(E+K)$$

$$g(\tilde{E}) \leq g_a(E)$$

$$X = X_0 \rightsquigarrow X_1 \to \cdots$$

<u>Def.</u> X is a minimal model surface if has no (-1)-curves.

Dim 3 and higher

Minimal Model Program

Reid, Mori, ...

Singularities

Hironaka

$\exists Y$ $\quad$ Y smooth

$$\begin{tikzcd} Y \arrow[d, "\text{generically 1-to-1}"] \\ X \end{tikzcd} \qquad K(Y)=K(X)$$

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4 classes of singularities

1) terminal
2) canonical
3) log-terminal
4) log-canonical

Adjunction for the resolution of singularities

$$\begin{array}{c} Y \\ \downarrow{\scriptstyle \pi} \\ X \end{array}$$

K_X is $\mathbb{Q}$-Cartier
(X is $\mathbb{Q}$-Gorenstein)

$$K_Y = \underline{\pi^* K_X} + \sum \underset{\in \mathbb{Q}}{a_i} E_i$$

where E_i are exceptional divisors ($\dim(\pi(E_i)) \leq d-2$)

a_j are <u>discrepancies</u>

terminal $\forall a_i > 0$

canonical $\forall a_i \geq 0$

log-terminal $\forall a_i > -1$

log-canonical $\forall a_i \geq -1$ (*)

(*) E_i simple normal crossings

Dim 2: terminal = smooth

canonical = du Val singularities

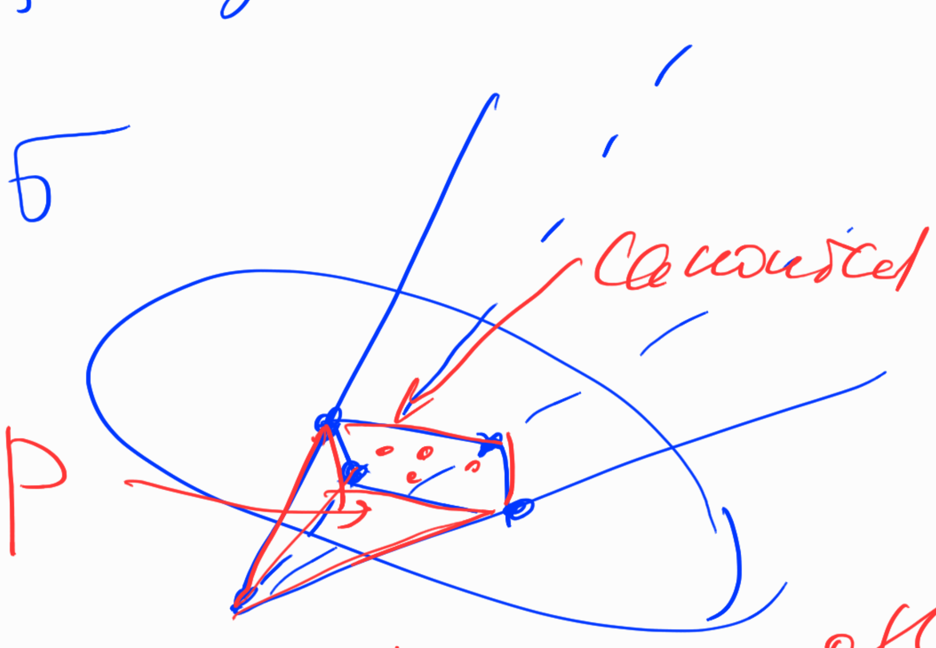
log-terminal = quotient

Any dim. quotient $\Rightarrow$ log-terminal

Toric varieties

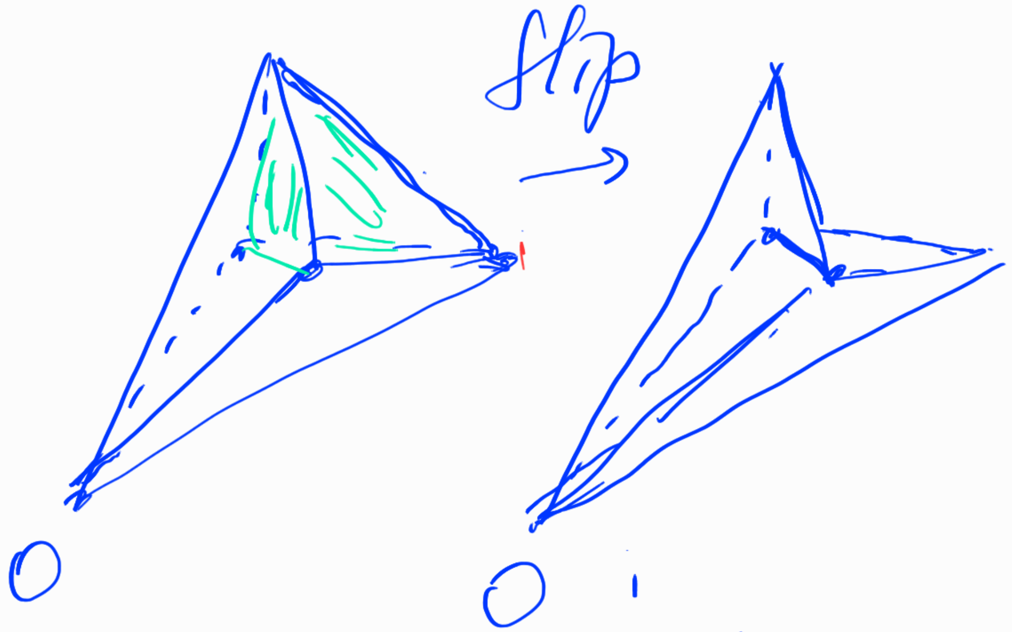
Every $\mathbb{Q}$-Gorenstein toric X is log-terminal

σ N dim 3



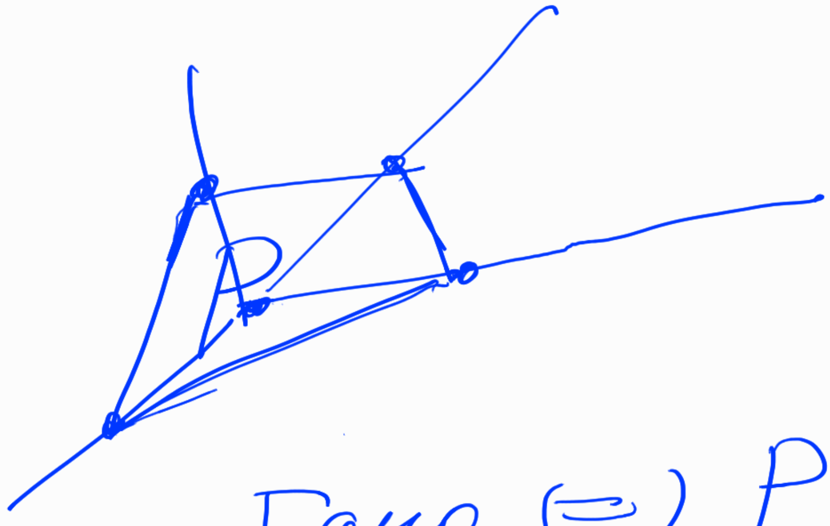
terminal $\Leftrightarrow$ no other lattice points on P

canonical: allow points on the face of P that does not contain 0.



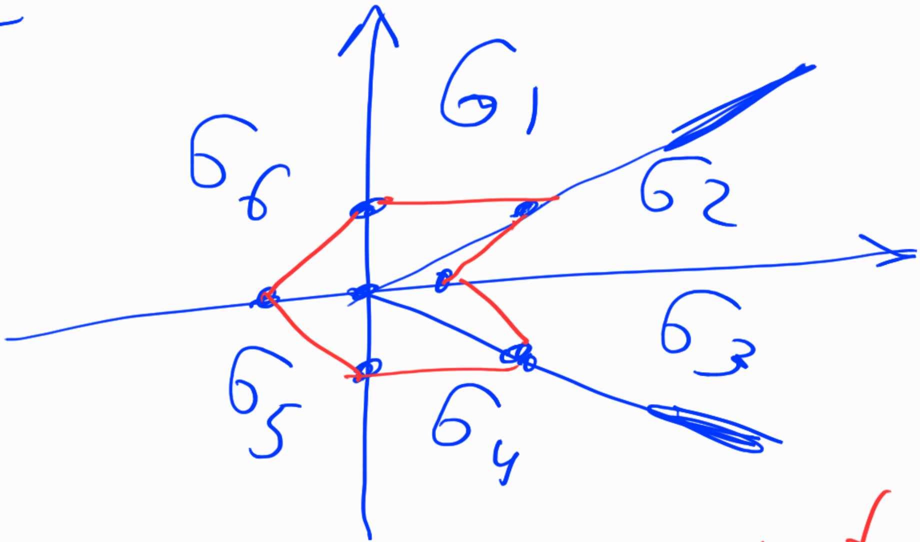
Def. X is a Fano variety if $-K_X$ is ample

Toric Fano varieties with terminal singularities:



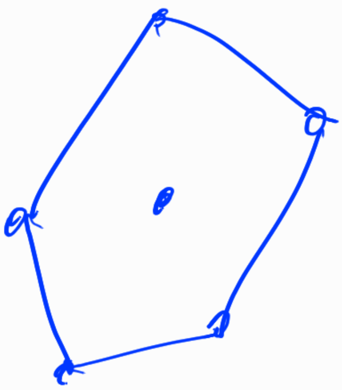
X_Σ is Fano $\iff$ P is convex

Ex. Not Fano, dim 2



not convex, so not Fano

Thm. In any dimension there are only finitely many toric Fano varieties with terminal singularities.



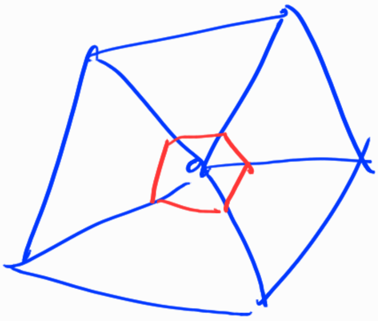
Hensley, Lagarias–Ziegler

A. Borisov – Lev Borisov

Moreover, $\forall$ dim $= d$,
$\forall \varepsilon > 0$ there are f. many
ε-log-terminal toric Fano
varieties in dimension d.

$\forall \varepsilon > 0$ only f. many
convex lattice polytopes P
containing $\vec{0}$, s.t.

$$\varepsilon \vec{P} \cap N = \{\vec{0}\}$$



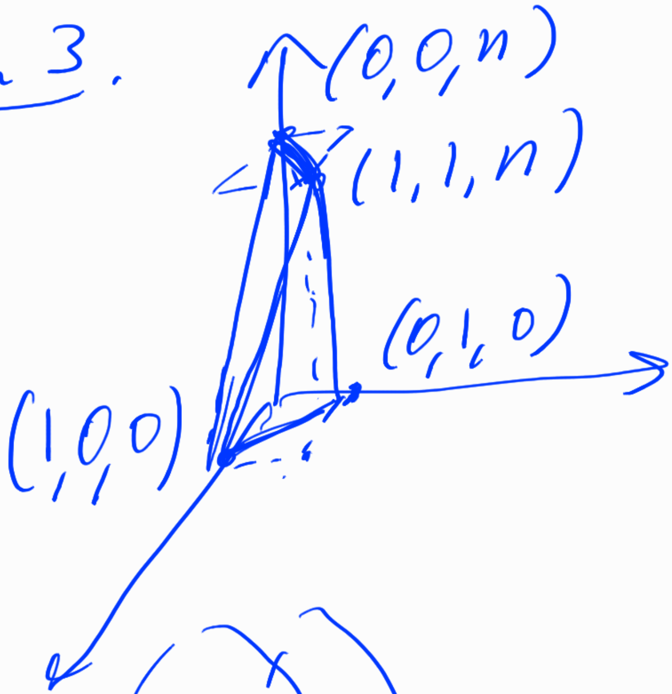
ε-log-terminal
$\Updownarrow$
$\forall a_i > -1 + \varepsilon$

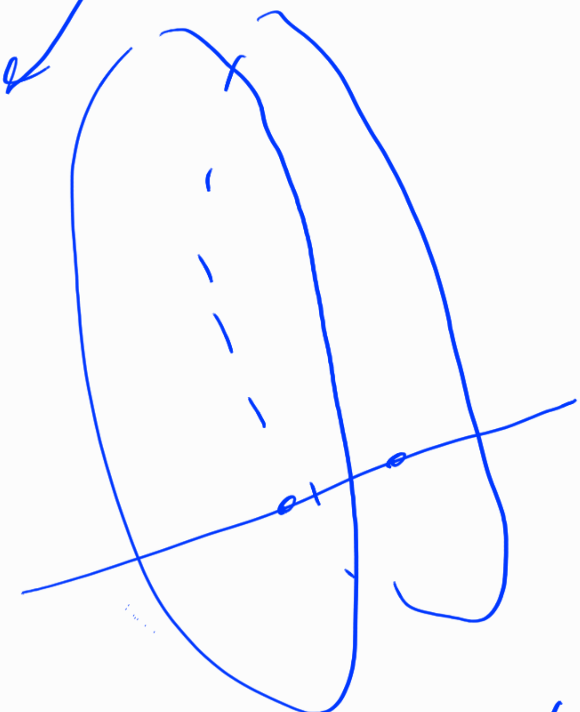
Conjecture (BAB Conjecture)
$\forall d$ $\forall \varepsilon$ only finitely
many families of Fano
varieties of dim d with
ε-logterminal singularities.

Caucher Birkar

Fields Medal

Dim 3.





MMM paper 1989

S. Mori - D. Morrison - I. Morrison

cyclic quotient singularities

in dim 4, prime index p