Minkowski Lemma

$\mathbb{Z}^n \subseteq \mathbb{R}^n$

$X \subset \mathbb{R}^n$

1) X convex
2) $(-X) = X$
3) $\mathrm{vol}(X) > 2^n$

Then $\exists\, \vec{x} \in X \cap \mathbb{Z}^n$, $\vec{x} \neq \vec{0}$

Remarks

1) 2^n is best possible:

$(-1,1)^n \subset \mathbb{R}^n$

has volume 2^n

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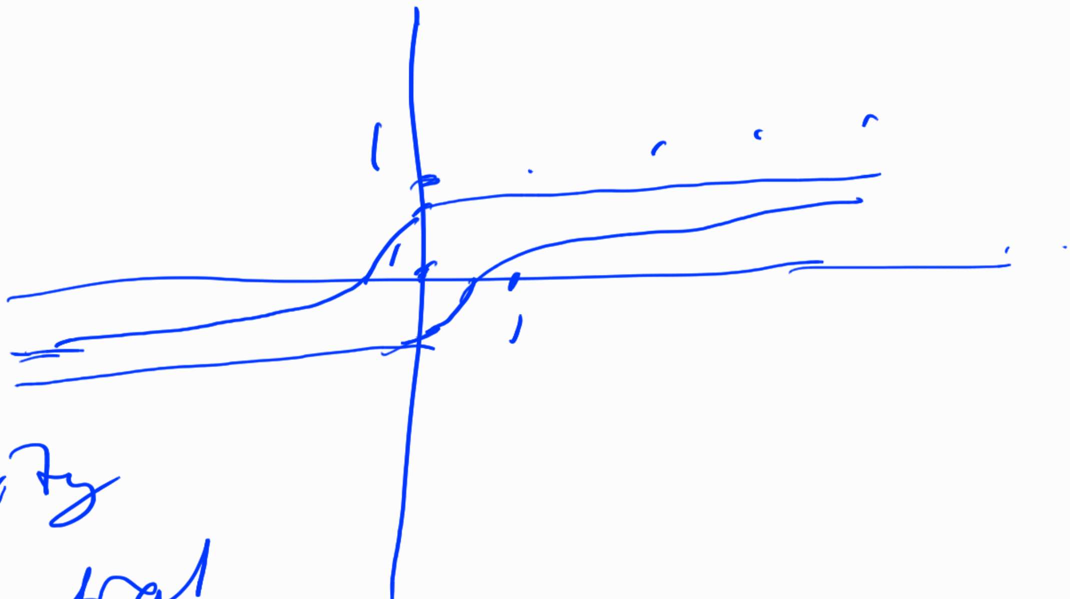
2) $L \subseteq \mathbb{R}^n$

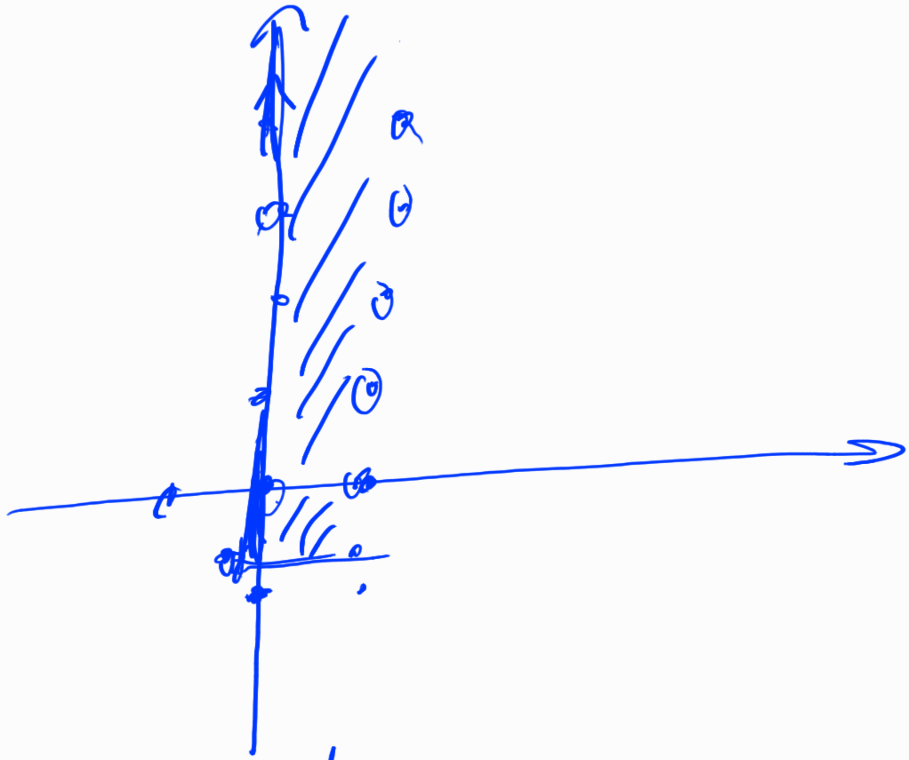
$\mathrm{vol}(X) > 2^n \cdot \mathrm{covol}(L)$

3)



Convexity
is essential

4)



$-X = X$
is essential

Proof. $\mathbb{R}^n$ is a union of shifts of the fundamental hypercube $[0,1)^n$
"reduce $\frac{1}{2}X$ modulo $\mathbb{Z}^n$":

$\forall \vec{x} \in \frac{1}{2}X \quad f(\vec{x}) = \{\vec{x}\} \in [0,1)^n$

The resulting set $\{ f(\vec{x}) \mid \vec{x} \in \frac{1}{2}X \}$ is in $[0,1)^n$

The volume of this is $\mathrm{vol}(\frac{1}{2}X)$ if there are no $\vec{x}, \vec{y} \in \frac{1}{2}X$ s.t. $\{\vec{x}\} = \{\vec{y}\}$

$$\mathrm{vol}(\tfrac{1}{2}X) = \frac{1}{2^n}\underline{\mathrm{vol}(X)} > 1$$

This is impossible so

$\exists\ \vec{x}, \vec{y} \in \frac{1}{2}X \quad \vec{x} \neq \vec{y}$

$$\vec{0} \neq \vec{x} - \vec{y} \in \mathbb{Z}^n$$

$$\shortparallel$$

$$\frac{1}{2}\vec{u} + \frac{1}{2}(-\vec{v})$$

$$\shortparallel$$

$$\frac{\vec{u} + (-\vec{v})}{2} \in X$$

$\vec{u}, \vec{v} \in X$ so $-\vec{v} \in X$, so

$\square$

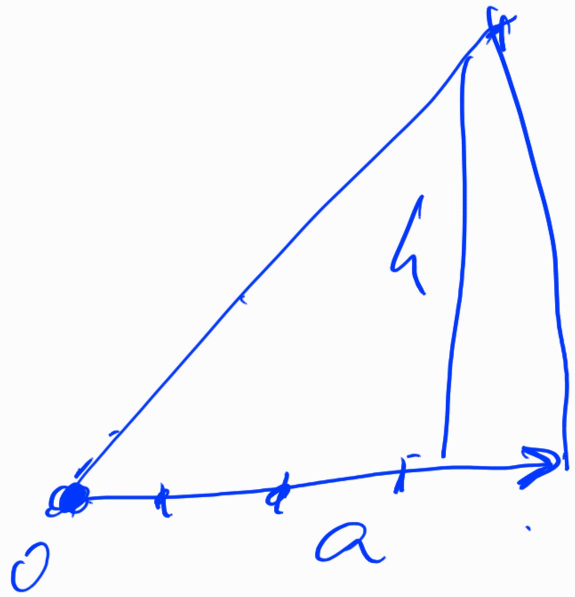
Goal for today

Fix dimension d.

<u>Thm.</u> There are only f. many lattice simplexes in $\mathbb{Z}^d$ that contain $\{\vec{0}\}$ and no other points in $\mathbb{Z}^d$ except vertices, up to action of $SL_d(\mathbb{Z})$

1) This is equivalent to having an upper bound on volume of such simplexes.

$d=2$

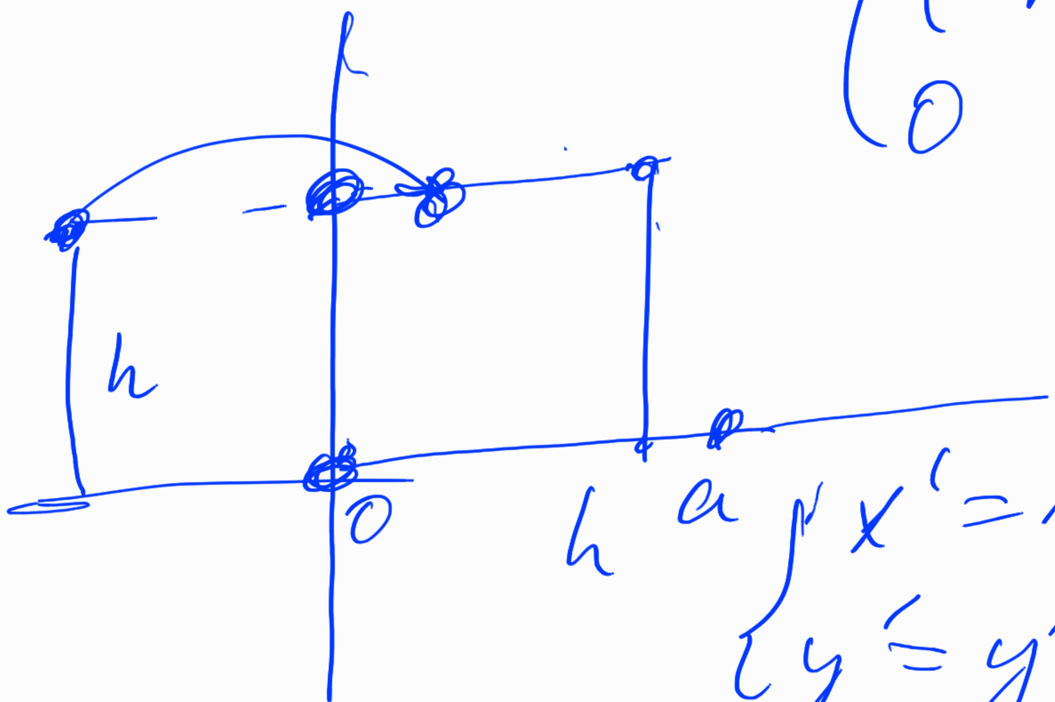


$$V = \frac{1}{2} h a$$

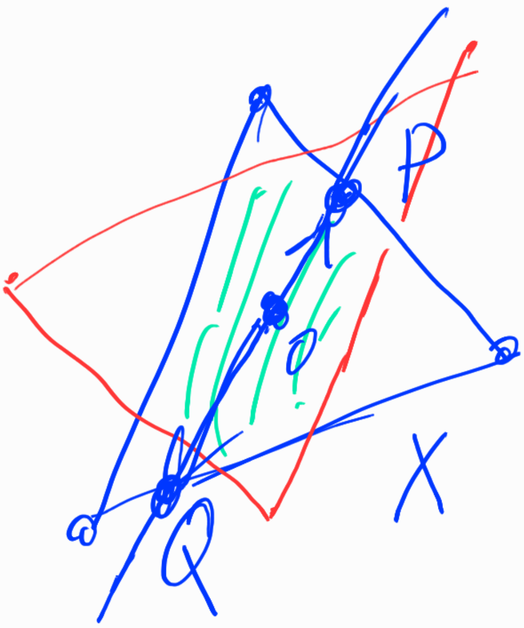
$a \le 2V$ bounded.

$h \le 2V$ bounded. ?

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$



$$\begin{cases} x' = x + ny \\ y = y' \end{cases}$$



$f: S^{d-1} \to \mathbb{R}$
ratio of lengths $\frac{|OP|}{|OQ|}$

$X \cap (-X)$ is convex
centrally symmetric.
By Minkowski Lemma,
if it has large enough
volume, it must have a
non-zero lattice point,
contradicts $X \cap \mathbb{Z}^n = \{\vec{0}, \text{vertices}\}$

It is enough to bound
$\max(f)$.

If $\max(f) = c > 1$

Then $\frac{1}{c}X \subseteq (-X) \cap X$

So $\underline{\text{vol}((-X) \cap X) \geq \frac{1}{c^d} \text{vol}(X)}$

Barycentric coordinates

$X = \vec{P}_0 \vec{P}_1, \dots \vec{P}_d$

$\forall \vec{x} \in X \quad \exists! \quad \vec{x} = \sum_{i=0}^{d} b_i \vec{P}_i$

s.t. $\begin{cases} b_i \geq 0 & \forall i \\ \sum_{i=0}^{d} b_i = 1 \end{cases}$

$\vec{o} \in X$

$\vec{o} = \frac{a_0}{h} \vec{P}_0 + \frac{a_1}{h} \vec{P}_1 + \dots + \frac{a_d}{h} \vec{P}_d$

$a_i > 0$, $\underline{\sum a_i = h}$

$c(X)$ only depends on $\{a_0, \dots, a_d\}$

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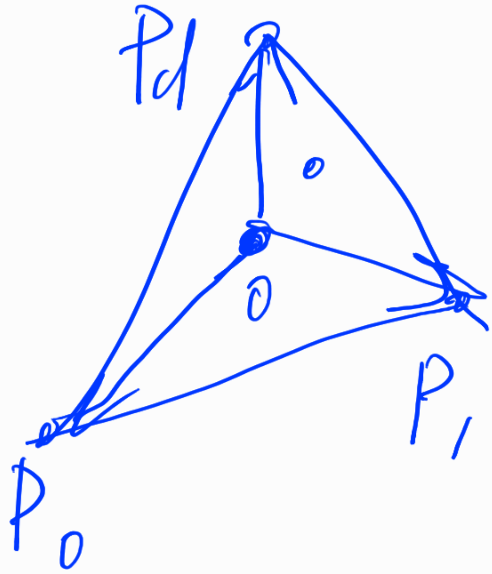
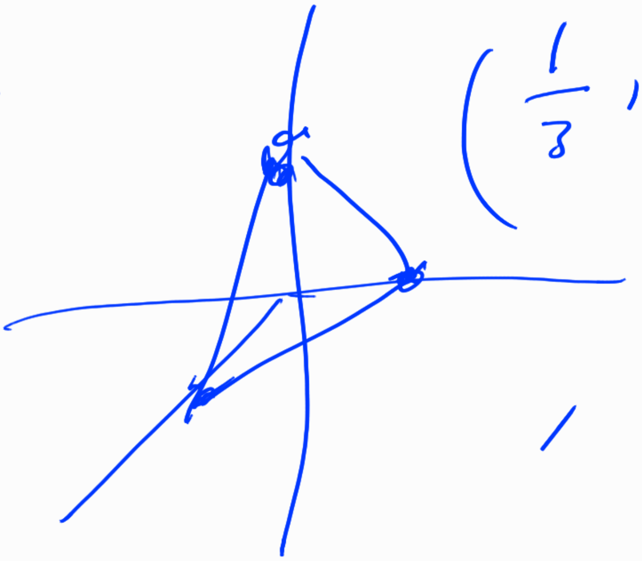
$\max(f)$

So we just need to show that there are only f. many $\{a_0 \dots a_d\}$ that there are no non-zero points in

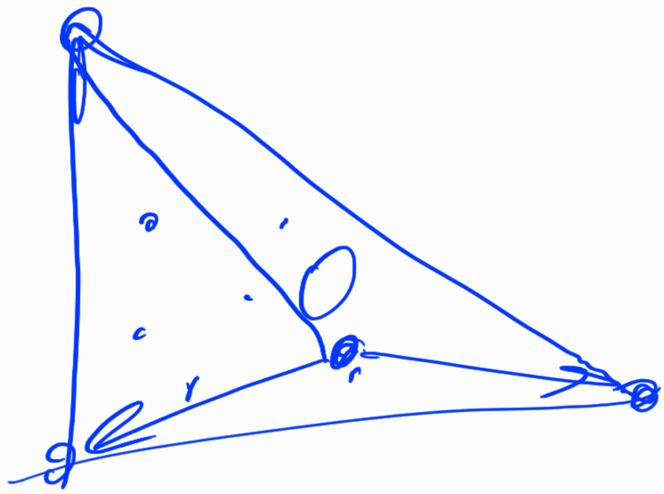
$$X \cap \left\{ \sum_{i=0}^{d} n_i \vec{P_i} \right\}$$



$\mathbb{CP}^2$ $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$



Consider all possible ∂
in a standard simplex X



s.t. the lattice
generated by it has
no points inside X.
Suppose there are
∞ many such O.

Take a limit point

$\exists\, mO$ close to O

$$O = \lim_{i\to\infty} O^{(i)} \qquad mO^{(i)} \to mO$$

<u>$m > 1$</u>