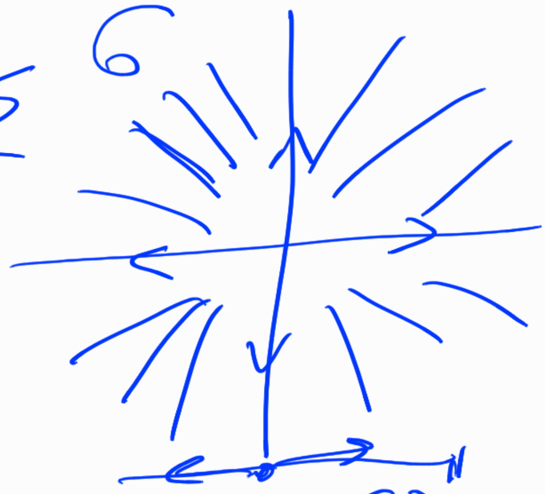
1) X_Σ



a) $X_\Sigma \cong \mathbb{C}P^1 \times \mathbb{C}P^1$

$\sigma = \langle (-1,0), (0,1) \rangle$

x_1, x_2

$\hat{\sigma} = \langle (-1,0), (0,1) \rangle$

$\hat{\sigma} \cap M = \langle (-1,0), (0,1) \rangle$

$m\mathrm{Spec}(\hat{\sigma} \cap M) = m\mathrm{Spec}\, \mathbb{C}[x_1^{-1}, x_2]$

$\mathbb{C}P^1$ is covered by two copies of $\mathbb{C}$:

$m\mathrm{Spec}(\mathbb{C}[x])$,

$m\mathrm{Spec}(\mathbb{C}[x^{-1}])$.

$\mathbb{CP}^1 \times \mathbb{CP}^1$ is covered by

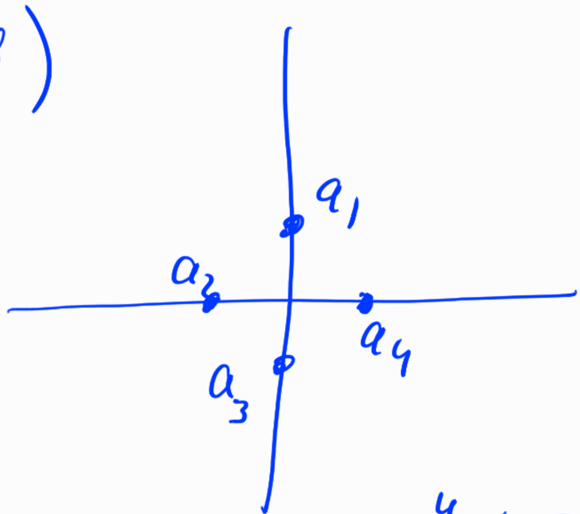
$m\mathrm{Spec}(\mathbb{C}[x_1, x_2])$,

$m\mathrm{Spec}(\mathbb{C}[x_1^{-1}, x_2])$,

$m\mathrm{Spec}(\mathbb{C}[x_1, x_2^{-1}])$

$m\mathrm{Spec}(\mathbb{C}[x_1^{-1}, x_2^{-1}])$

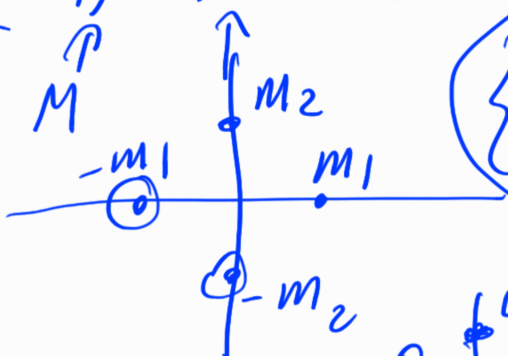
b)



$$\mathrm{Pic}(X_\Sigma) = \mathbb{Z}^4 / \underbrace{\mathbb{Z}^2}_{M}$$

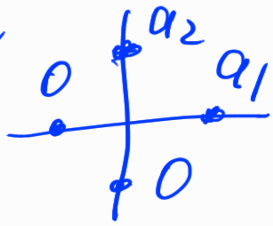
$(m_1, m_2) : (u_1, u_2) \mapsto u_1 m_1 + u_2 m_2$

M

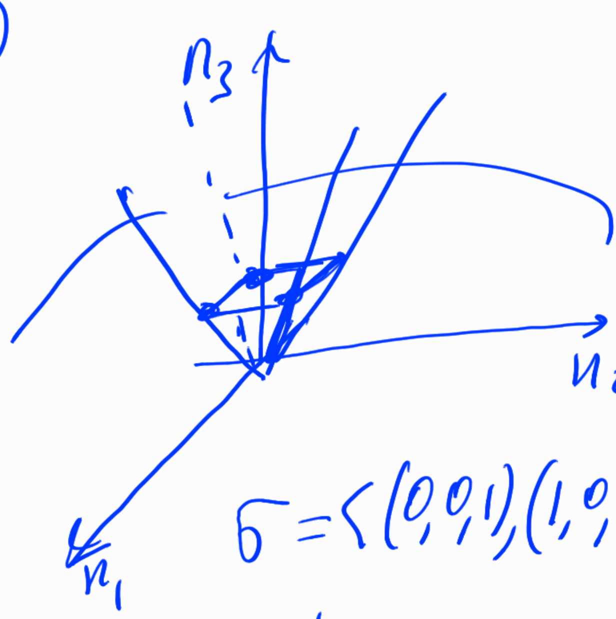


$$\begin{cases} a_1 + a_3 = 0 \\ a_2 + a_4 = 0 \end{cases}$$

<u>Answer:</u> $\mathbb{Z}^2$

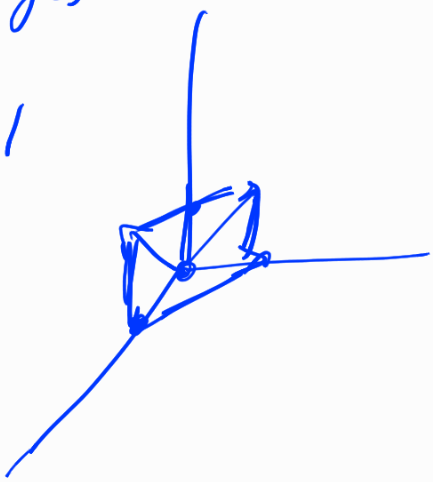


2



$\sigma = \langle (0,0,1), (1,0,1), (0,1,1), (1,1,1) \rangle$

$f = n_3$ takes values 1 on all 4 generators of the rays.



$n_1 + n_2 = 1$

$(1,0,0)$

$(0,1,0)$

$(1,0,1)$

$(0,1,1)$

X, X' toric

$\Sigma \quad \Sigma'$

$N \quad N'$

$X \times X'$

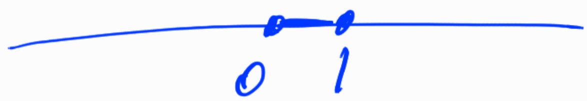
$N \oplus N'$

$\sigma \times \sigma' = \{(x,y) \mid x \in \sigma, x' \in \sigma'\}$

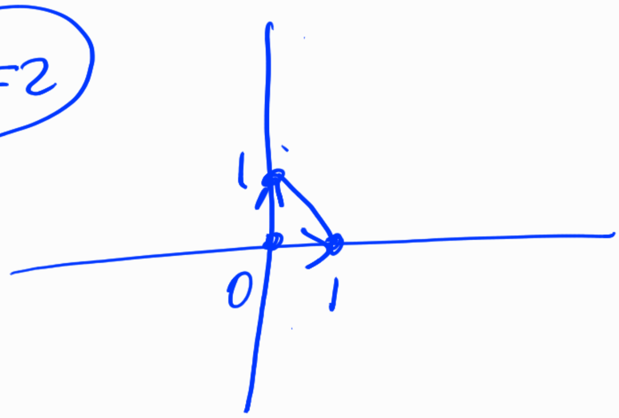
$\begin{array}{cc} \sigma & \sigma' \\ \Uparrow & \Uparrow \\ \Sigma & \Sigma' \end{array}$

Lattice-free lattice simplices / simplexes

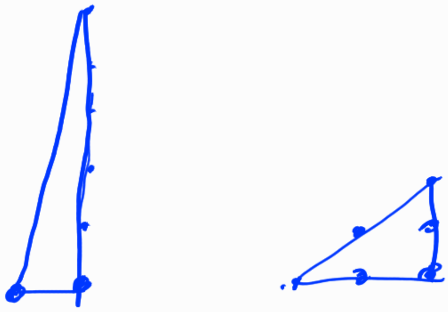
d=1



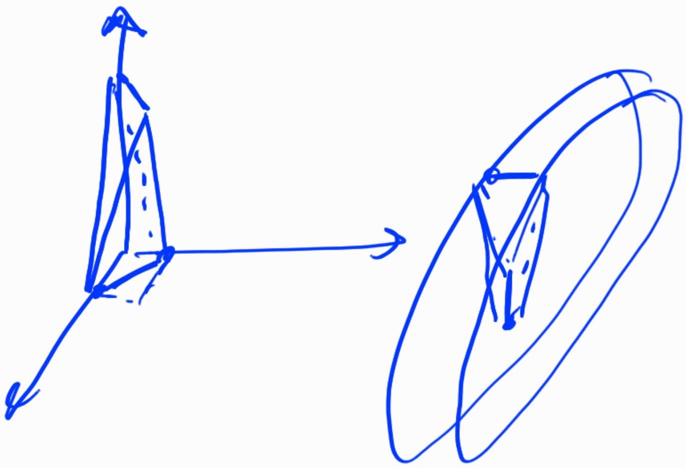
d=2



Remark. If we allow points on the boundary:



<u>dim 3</u>



<u>Thm.</u> Every lattice-free simplex X in dim 3 has width 1:

$\exists f: \mathbb{Z}^3 \to \mathbb{Z}$ <u>linear</u>

s.t. $f(X)$ is a segment of length 1

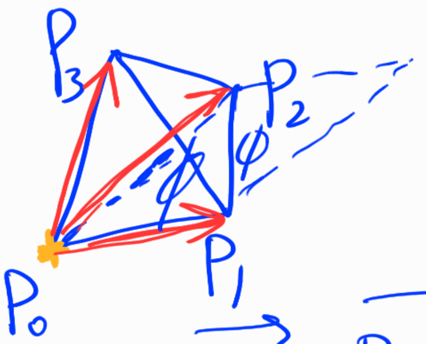
<u>Def.</u> X lattice polytope compact.

$$\mathrm{width}(X) = \min_{f: \mathbb{Z}^d \to \mathbb{Z}} |f(X)|$$

$|f(x)|$ length of the range.

Idea #1

$d=3$



$L = \langle \overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \overrightarrow{P_0P_3} \rangle$
("$\langle\ \rangle$" as abelian group)

$L \subseteq \mathbb{Z}^d$ finite index

Lemma. $\mathbb{Z}^3/L$ is cyclic (or trivial) if X is lattice-free.

Proof. If $\mathbb{Z}^3/L$ is not cyclic, then $\exists$ prime p s.t. $\mathbb{Z}^3/L$ contains a subgroup $\cong (\mathbb{Z}/p\mathbb{Z})^2$

Consider the generators $\vec{x}, \vec{x}' \in \mathbb{Z}^n$ $px, px' \in L$

Write them in coordinates of L:

$$\vec{x} = \frac{1}{p} a_1 \overrightarrow{P_0P_1} + \frac{1}{p} a_2 \overrightarrow{P_0P_2} + \frac{1}{p} a_3 \overrightarrow{P_0P_3}$$

$\vec{x}' = \frac{1}{p}(a_1', a_2', a_3')$ in the basis $\{\overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \overrightarrow{P_0P_3}\}$

a_i, a_i' modulo p

$$\begin{cases} \vec{x} = \frac{1}{p}(a_1, a_2, a_3) \\ \vec{x}' = \frac{1}{p}(a_1', a_2', a_3') \end{cases}$$

claim

$\underbrace{a_1 \neq 0}$ $\underline{\left(0, \frac{a_2}{p}, \frac{a_3}{p}\right)}$

in the plane

of $\overrightarrow{P_0P_2}, \overrightarrow{P_0P_3}$

$P_0P_2P_3$ is a latticefree

symplex, so the

plane of $P_0P_2P_3$ cannot

contain points in $L \setminus \mathbb{Z}^3$.

$$x = \frac{1}{p}(\cancel{a_1}, a_2, a_3)$$

$$x' = \frac{1}{p}(\cancel{a_1'}, a_2', a_3')$$

$$\underset{\neq 0}{x - x'} = \frac{1}{p}(0, a_2 - a_2', a_3 - a_3')$$

$x - x' \in L \setminus \mathbb{Z}^3$, contradiction $\square$

$\mathbb{Z}^3/L$ is cyclic,
so it is generated
by $Q = \frac{1}{r}(a_1, a_2, a_3)$
in the basis
$\overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \overrightarrow{P_0P_3}$

$r = |\mathbb{Z}^3/L|$