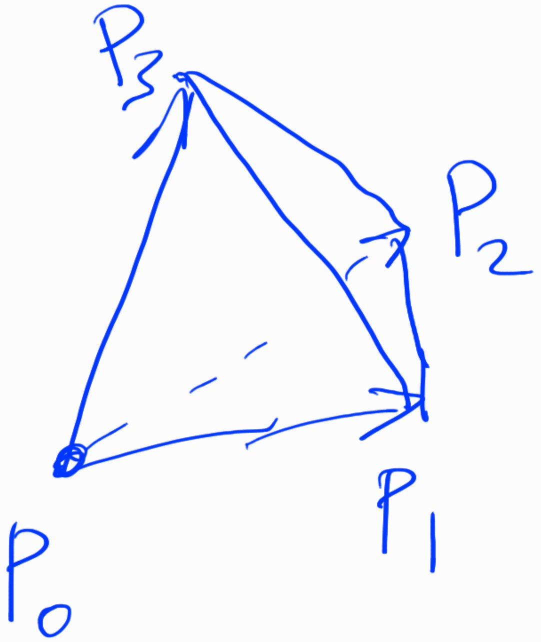
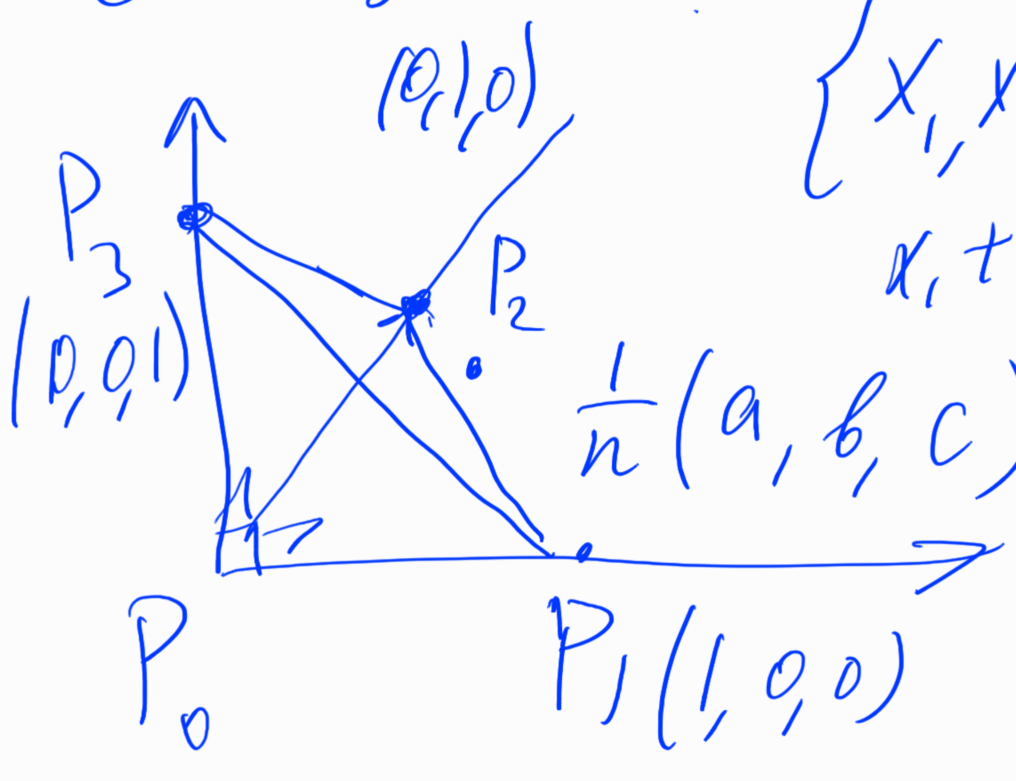


$$L = \langle \overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \overrightarrow{P_0P_3} \rangle$$

$\mathbb{Z}^3/L$ is cyclic

Change the viewpoint



$$\left\{ \begin{array}{l} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 + x_3 \leq 1 \end{array} \right\}$$

$\frac{1}{n}(a, b, c)$

$\gcd(a,b,c) = 1$

Lemma. $\gcd(a,n)=1$
(and same for b, c, $a+b+c$)

Proof. If $\gcd(a,n)=d>1$

then $n=n_1\cdot d$

$$n_1\cdot\left(\frac{1}{n}(a,b,c)\right)=\frac{1}{d}(a,b,c)$$

$$\overset{\cap}{N} \qquad \frac{a}{d}\in\mathbb{Z}$$

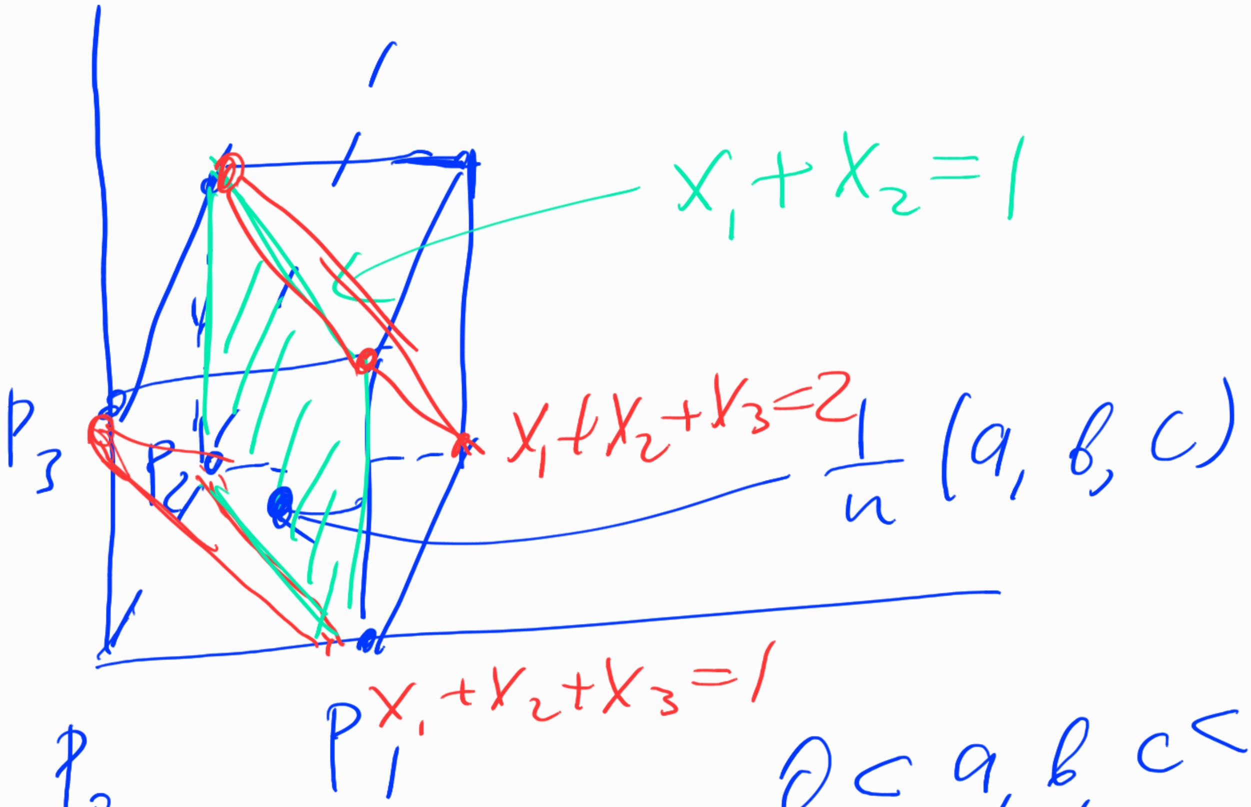
Subtracting $\left(\frac{a}{d},0,0\right)$

we get a point $\frac{1}{d}(0,b,c)$

This is in $N\setminus\mathbb{Z}^3$,

on the plane $x_1=0$.

Impossible.



$0<a,b,c<n$

$\frac{1}{n}(a,b,c)$

$N\cap P_0P_1P_2P_3=\{P_0,P_1,P_2,P_3\}$

$\Updownarrow$

$\forall k=1,2,\dots,n-1$

$\left\{\frac{ka}{n}\right\}+\left\{\frac{kb}{n}\right\}+\left\{\frac{kc}{n}\right\}>1$

If $\frac{a}{n}+\frac{b}{n}=1$, then $\forall k\in\{1,\dots,n-1\}$

$\left\{\frac{ka}{n}\right\}+\left\{\frac{kb}{n}\right\}=1$

Then $\underbrace{\left\{\frac{ka}{n}\right\}+\left\{\frac{kb}{n}\right\}+\left\{\frac{kc}{n}\right\}}_{1}>1$

$\forall k\in\{1,2,\dots,n-1\}$

<u>Thm</u> (White, Morrison-Stevens)

(This proof is based on the idea of Oesterlé)

If $\frac{1}{n}(a,b,c)$ generates a lattice-free simplex then $a+b=n$ or $a+c=n$ or $b+c=n$.

<u>Proof.</u> Introduce d

$d = -(a+b+c)$

So that $a+b+c+d=0$.

Claim. $\forall\, k = 1, 2, \dots, n-1$

$$\left\{\frac{k}{n}a\right\} + \left\{\frac{k}{n}b\right\} + \left\{\frac{k}{n}c\right\} + \left\{\frac{k}{n}d\right\} = 2$$

Proof of claim.

$$1 < \left\{\frac{k}{n}a\right\} + \left\{\frac{k}{n}b\right\} + \left\{\frac{k}{n}c\right\} < 2$$

If $\left\{\frac{k}{n}a\right\} + \left\{\frac{k}{n}b\right\} + \left\{\frac{k}{n}c\right\} \geq 2$ (< 3)

then for $k_1 = n - k$:

$$\left\{\frac{k_1 a}{n}\right\} = \left\{a - \frac{ka}{n}\right\} = 1 - \left\{\frac{ka}{n}\right\}$$

$$\left\{\frac{k_1 a}{n}\right\} + \left\{\frac{k_1 b}{n}\right\} + \left\{\frac{k_1 c}{n}\right\} =$$

$$3 - \underbrace{\left(\left\{\frac{ka}{n}\right\} + \left\{\frac{kb}{n}\right\} + \left\{\frac{kc}{n}\right\}\right)}_{\geq 2} \leq 1$$

impossible.

So $1 < \left\{\frac{ka}{n}\right\} + \left\{\frac{kb}{n}\right\} + \left\{\frac{kc}{n}\right\} + \left\{\frac{kd}{n}\right\} < 3$

$\mathbb{Z}$ $\quad$ So $= 2$ $\square$

$$\left\{\frac{ka}{n}\right\} \quad k=1,2,\ldots,n-1$$

$\gcd(a,n)=1$

As a set: $\left\{\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n}\right\}$

In what follows ζ is in $\mathbb{C}$

s.t. $\zeta^n=1$.

$$\sum_{k=1}^{n-1}\left(\left\{\frac{ka}{n}\right\}+\left\{\frac{kb}{n}\right\}+\ldots+\left\{\frac{kd}{n}\right\}\right)\zeta^k=\sum_{k=1}^{n-1}?\zeta^k$$

$$\sum_{k=1}^{n-1}\left\{\frac{ka}{n}\right\}\zeta^k=\underline{\sum_{k=1}^{n-1}\frac{k}{n}\zeta^{a'k}}$$

$\gcd(a,n)=1 \quad \exists\, a' \quad a'a\equiv 1 \bmod n$

$a'(ak)\equiv k \pmod n$

$\left\{\frac{ka}{n}\right\}=\frac{k'}{n} \iff ak\equiv k' \pmod n$

$k\equiv k'a' \pmod n$

$\zeta^k=\zeta^{a'k'}$

Lemma. $\sum_{k=1}^{n-1} k x^k = \frac{n}{x-1}$

$x \neq 1$

$x^n = 1$

$$\| $$

$$x \cdot \left(1 + \dots + x^{n-1}\right)'$$

$$x \cdot \left(\frac{x^n - 1}{x-1}\right)' = x \frac{n x^{n-1} \cancel{(x-1)} - \cancel{(x^n - 1)}}{(x-1)^{\cancel{2}}}$$

(with $x^n - 1 = 0$ written above the struck term)

At $x^n = 1$, $x \neq 1$

$$\| $$

$$\frac{n x^n}{x-1} = \frac{n}{x-1}$$

$\forall \zeta \neq 1 \quad \zeta^n = 1$

$$\frac{1}{\zeta^{a'} - 1} + \frac{1}{\zeta^{b'} - 1} + \frac{1}{\zeta^{c'} - 1} + \frac{1}{\zeta^{d'} - 1} = \underline{2(-1)}$$

$$\left(\frac{1}{\zeta^{a'} - 1} + \frac{1}{\zeta^{b'} - 1} + 1\right) + \left(\frac{1}{\zeta^{c'} - 1} + \frac{1}{\zeta^{d'} - 1} + 1\right) = 0$$

$$\frac{\zeta^{b'}-1+\zeta^{a'}-1+\left(\zeta^{a'+b'}-\zeta^{a'}-\zeta^{b'}+1\right)}{\left(\zeta^{a'}-1\right)\left(\zeta^{b'}-1\right)}$$

$$\frac{\zeta^{a'+b'}-1}{\left(\zeta^{a'}-1\right)\left(\zeta^{b'}-1\right)}+\frac{\zeta^{c'+d'}-1}{\left(\zeta^{c'}-1\right)\left(\zeta^{d'}-1\right)}=0$$

$\dots$

$\forall \zeta \neq 1$ $\quad \left(\zeta^{a'+b'}-1\right)\left(\zeta^{a'+c'}-1\right)\left(\zeta^{a'+d'}-1\right)=0$

$\zeta^n=1$

This can only happen

if $\begin{cases} a'+b' \equiv 0 \pmod{n} \\ a'+c' \equiv 0 \pmod{n} \\ a'+d' \equiv 0 \pmod{n} \end{cases}$

If $a'+b' \equiv 0 \pmod{n}$

$a' \equiv -b' \pmod{n}$

$a \equiv -b \pmod{n}$

$$a+b \equiv 0 \pmod{n}$$