$(a, b, c, d) \bmod n$

$\gcd(a, n) = 1$

$\gcd(b, n) = 1$

$\gcd(c, n) = 1$

$\gcd(d, n) = 1$

$a + b + c + d = 0$

$\forall k = 1, 2, \ldots, n-1$

$$\left\{\frac{ka}{n}\right\} + \left\{\frac{kb}{n}\right\} + \left\{\frac{kc}{n}\right\} + \left\{\frac{kd}{n}\right\} = 2$$

Thm. All such (a, b, c, d) are:

$$\begin{cases} a + b = 0 \\ a + c = 0 \\ a + d = 0 \end{cases} \pmod n$$

$a'a \equiv 1 \pmod n$

$\zeta^n = 1$

$$\sum_{k=1}^{n-1} \left\{\frac{ka}{n}\right\} \zeta^k = \sum_{k=1}^{n-1} \frac{k}{n} \zeta^{a'k}$$

$\underline{\zeta \neq 1}$

$$\frac{1}{\zeta^{a'}-1}+\frac{1}{\zeta^{b'}-1}+\frac{1}{\zeta^{c'}-1}+\frac{1}{\zeta^{d'}-1}=-2$$

$$\frac{\zeta^{a'+b'}-1}{(\zeta^{a'}-1)(\zeta^{b'}-1)}+\frac{\zeta^{c'+d'}-1}{(\zeta^{c'}-1)(\zeta^{d'}-1)}=0$$

$$\frac{2\zeta^{a'+b'+c'+d'}-2}{-(\zeta^{a'+b'+c'}+\zeta^{a'+b'+d'}+\zeta^{a'+c'+d'}+\zeta^{b'+c'+d'})+\underline{(\zeta^{a'}+\zeta^{b'}+\zeta^{c'}+\zeta^{d'})}}=0$$

$\underline{\forall \zeta \neq 1}$, $\zeta^n=1$

can be removed

Add this up for all ζ, $\zeta^n=1$

$$\sum_{\zeta^n=1} \zeta^m = \begin{cases} 0, & n \nmid m \\ n, & n \mid m \end{cases}$$

$$2\begin{cases}0\\n\end{cases}-2n-\begin{cases}0\\n\end{cases}-\begin{cases}0\\n\end{cases}-\begin{cases}0\\n\end{cases}-\begin{cases}0\\n\end{cases}=0$$

Must be n

$$a'+b'+c'+d' \equiv 0 \mod n$$

$$-(\zeta^{a'+b'+c'} + \zeta^{a'+b'+d'} + \zeta^{a'+c'+d'} + \zeta^{b'+c'+d'})$$

$$+(\zeta^{a'}+\zeta^{b'}+\zeta^{c'}+d') = 0$$

$$\zeta^{-a'}(\zeta^{a'+b'}-1)(\zeta^{a'+c'}-1)(\zeta^{a'+d'}-1) =$$

$$= \zeta^{-a'}(\zeta^{3a'+b'+c'+d'} - \zeta^{2a'+b'+c'} - \zeta^{2a'+b'+d'} - \zeta^{2a'+c'+d'}$$

$$+ \zeta^{a'+b'} + \zeta^{a'+c'} + \zeta^{a'+d'} - 1)$$

$$= \zeta^{a'} - \zeta^{a'+b'+c'} - \zeta^{a'+b'+d'} - \zeta^{a'+c'+d'}$$

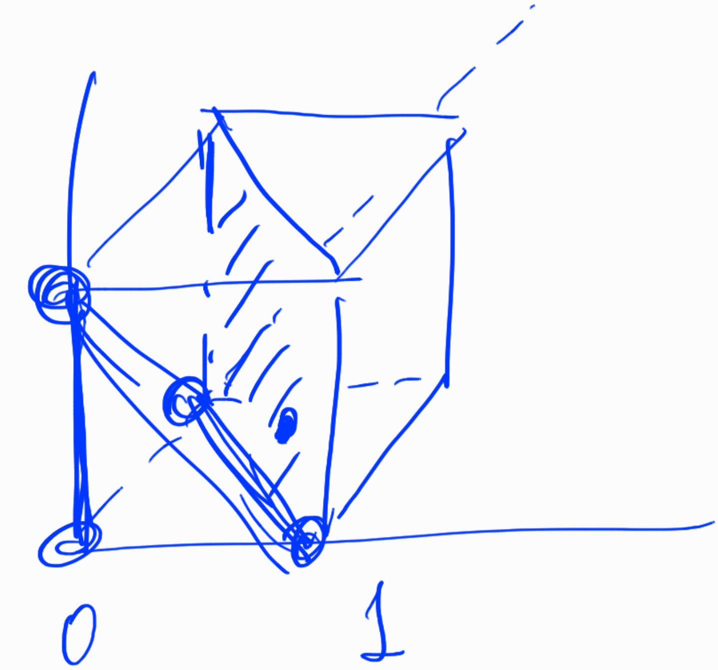
$$+ \zeta^{b'}+\zeta^{c'}+\zeta^{d'} - \zeta^{b'+c'+d'}$$

So $\forall \zeta \quad \zeta^n = 1$

$$(\zeta^{a'+b'}-1)(\zeta^{a'+c'}-1)(\zeta^{a'+d'}-1)=0$$

Take ζ primitive n-th root of unity (e.g. $e^{2\pi i/n}$)

then $a'+b'$ or $a'+c'$ or $a'+d' = 0 \pmod n$ ∎



Thm Lattice ~~polytope~~ polyhedron P (convex)
If P is lattice-free, then it has width 1.

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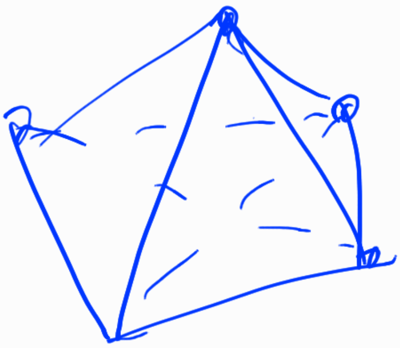
dimension 4

Mori-Morrison-Morrison 1988
Cyclic quotient singularities
in dimension 4 of prime
index p.

$P_0 P_1 P_2 P_3 P_4$



$N = \langle P_0 P_i, \ i = 1,2,3,4 \rangle$

$[N : \mathbb{Z}^4] \cong \mu_p$

Barile, Bernardi, A. B, Kantor

≈ 2015

used computer to calculate these.

Results:

two 2-parameter "families"

29 1-parameter "families"

many additional ones,

for $p < 421$

$\frac{1}{p}(a, b, c, d)$

$\forall k = 1, 2, \ldots, p-1$

$\left\{\frac{ka}{p}\right\} + \left\{\frac{kb}{p}\right\} + \left\{\frac{kc}{p}\right\} + \left\{\frac{kd}{p}\right\} > 1$

Add one more: e

so that $a + b + c + d + e = 0$

$\left\{\frac{ka}{p}\right\} + \ldots + \left\{\frac{ke}{p}\right\} \in \{2, 3\}$

<u>Results:</u>

1) $\underline{(\alpha, -\alpha}, \beta, \gamma, -\beta-\gamma)$

$\left\{\frac{k\alpha}{p}\right\} + \left\{-\frac{k\alpha}{p}\right\} = 1 \quad \forall k \not\equiv 0 \bmod p$

2) $(2\alpha, 2\beta, -\alpha, -\beta, -(\alpha+\beta))$

3) 29 stable quintuples:

e.g. $(1, 9, -2, -3, -5)$

$\forall p > 9$

$\forall k = 1, 2, \dots, p-1$

$$\left\{\frac{k}{p}\right\} + \left\{\frac{9k}{p}\right\} + \left\{\frac{-2k}{p}\right\} + \left\{\frac{-3k}{p}\right\} + \left\{\frac{-5k}{p}\right\} \in \{1, 3\}$$

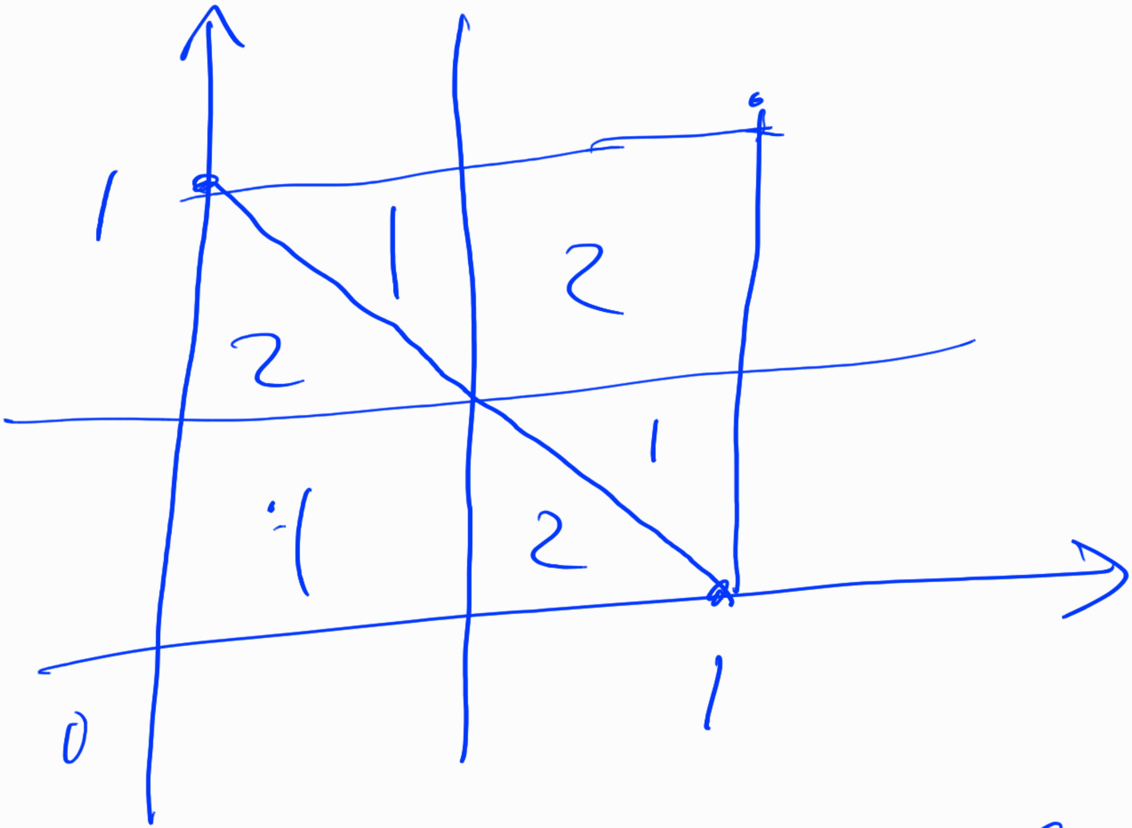
Lemma.

$$\left\{\frac{2\alpha k}{p}\right\} + \left\{\frac{2\beta k}{p}\right\} + \left\{\frac{-\alpha k}{p}\right\} + \left\{\frac{-\beta k}{p}\right\} + \left\{\frac{-(\alpha+\beta)k}{p}\right\} \in \{1, 3\}$$

Proof.

$\frac{\alpha k}{p} = x \qquad \frac{\beta k}{p} = y$

$$\{2x\} + \{2y\} + \{-x\} + \{-y\} + \{-(x+y)\} \in \{1, 2\}$$



$$-\{2x\}-\{2y\}-\underset{(-1)}{\{-x\}}-\underset{(-1)}{\{-y\}}-\underset{(-1)}{\{-(x+y)\}}$$

1989 Sankaran proved that the list of 29 stable quintuples is complete.

1988(?) J. Lawrence

2018 Santos + ...