James Lawrence ≈ 1988

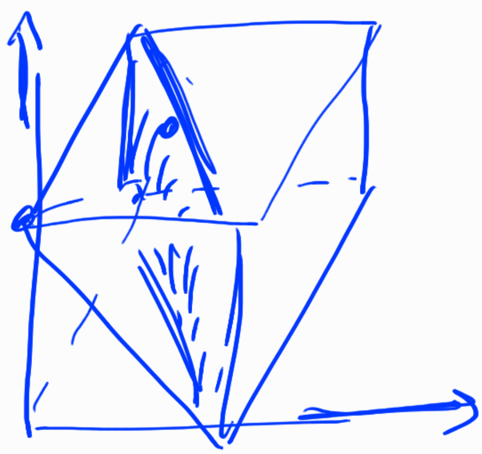
Theorem 1.

Suppose $T^n = (\mathbb{R}/\mathbb{Z})^n$ is the n-dimensional torus.

Suppose $V \subseteq T^n$ is a subset which is

1) closed
2) closed under multiplication by n $\forall n \in \mathbb{Z}$

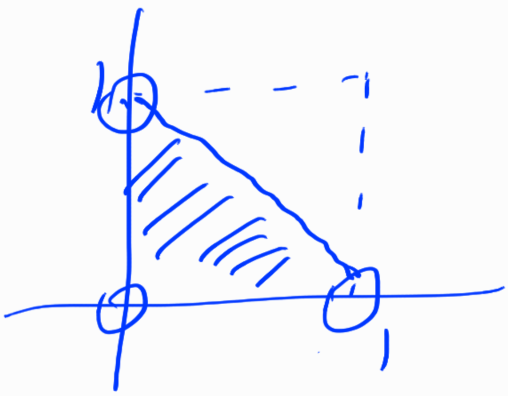
Then V is a finite union of closed subgroups of T^n.



Theorem 2.

Suppose $U \subset T^n$ is a "full set": every closed subgroup of T^n either has empty intersection with U or it contains a relatively open (in the subgroup) set. Then the set of all closed subgroups of T that avoid U has finitely many maximal elements.

Ex.



Closed subgroups of $\mathbb{R}^n$

Thm. Every closed subgroup G of $\mathbb{R}^n$ is of the form:

$$\left\{ \sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=k+1}^{l} m_i \vec{v}_l \right\}$$

where $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_l\}$ are linearly independent,

$a_i \in \mathbb{R}$, $m_i \in \mathbb{Z}$ $\quad (l \leq n)$

Note: if the subgroup is discrete, then $k = 0$.

Proof. I will consider

$$W = \{ \vec{v} \in \mathbb{R}^n \mid \forall r \in \mathbb{R} \;\; r\vec{v} \in G \}$$

0

Lemma. $W = \{\vec{0}\} \cup \{\vec{v}\,|$

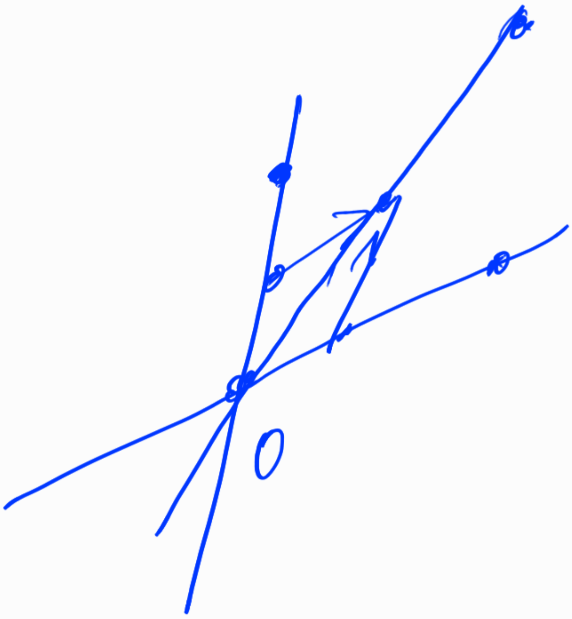
$\exists \{\vec{g}_1, \dots, g_m, \dots\}$ in G.

s.t. 1) $|g_m| \to 0$

2) $\frac{g_m}{|g_m|} \to \frac{\vec{v}}{|\vec{v}|}$ $\}$

$\forall \alpha > 0$ $\exists g \in G$
$\forall \varepsilon > 0$ $|g| < \varepsilon$, $\left|\frac{\vec{g}}{|\vec{g}|} - \frac{\vec{v}}{|\vec{v}|}\right| < \alpha$

Then by looking at multiples of g, we can get an element in G arbitrarily close to $\vec{v}$, so $\vec{v} \in G$. And any $r\vec{v} \in G$.



$W \cong \mathbb{R}^k$

Consider $\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^n/W$

$$\mathbb{R}^n/W \cong \mathbb{R}^{n-k}$$

$\varphi(G)$ is a closed subgroup of $\mathbb{R}^{n-k}$.

<u>Claim.</u> $\underline{\varphi(G)}$ is discrete.

If $\vec{0} = \lim\limits_{m\to\infty} g_m$, $g_m \in \varphi(G)$, $\bar{g}_m \neq \vec{0}$

we could find a subsequence

s.t. $\dfrac{\bar{g}_m}{|\bar{g}_m|} \longrightarrow \vec{y} \in \mathbb{R}^{n-k}$

$\vec{y} \neq \vec{0}$

Then $\forall r \in \mathbb{R}$ $r\vec{y} \in \varphi(G)$

Take $\vec{v} \in \mathbb{R}^n$ s.t. $\varphi(\vec{v}) = \vec{y}$

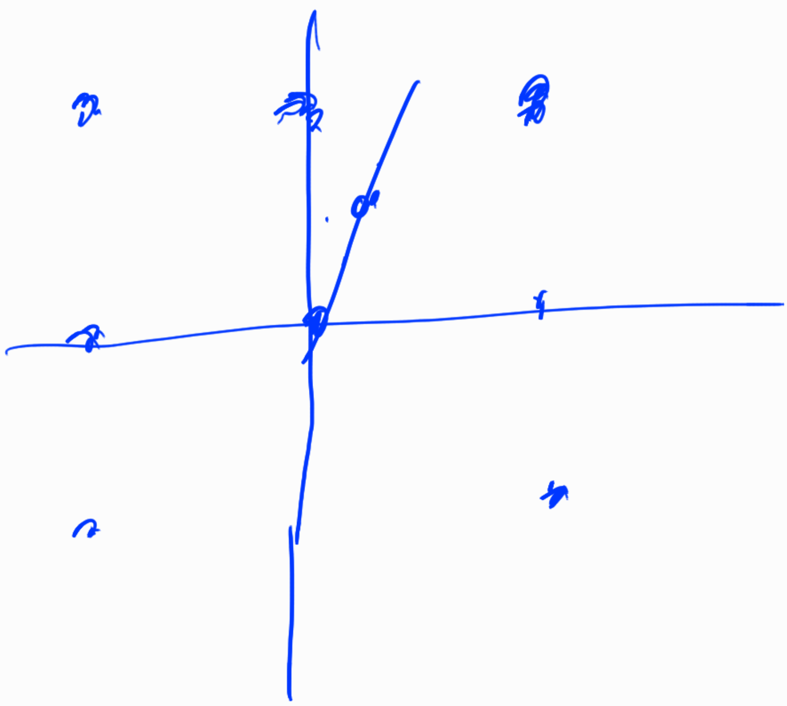
Then $\forall r \in \mathbb{R}$ $r\vec{v} \in G$:

$\varphi(r\vec{v}) = \varphi(g_r)$

$r\vec{v} - g_r \in W$

$r\vec{v} \in G$,

So $\vec{v} \in W$, but $\vec{v} \neq \vec{0}$,



Minkowski's theory of successive minima

Setup:

$\mathbb{Z}^n \subset \mathbb{R}^n$

$X \subset \mathbb{R}^n$ is centrally symmetric, convex, bounded, $vol(X) > 0$.

We can define a norm of $\mathbb{R}^n$ using this X.

$$X = \{\vec{v} \in \mathbb{R}^n \mid |\vec{v}| \leq 1\}$$

$$|\vec{v}| = \inf\{\lambda \in \mathbb{R}_{\geq 0} \mid \lambda X \ni \vec{v}\}$$

Why is $|\vec{v}|$ a norm?

Suppose $\begin{cases} \|\vec{v}_1\| = \lambda_1 \\ |\vec{v}_2| = \lambda_2 \end{cases}$

$$\begin{cases} \lambda_1 X \ni \vec{v}_1 \\ \lambda_2 X \ni \vec{v}_2 \end{cases}$$

$$\frac{1}{\lambda_1}\vec{v}_1 \in X \qquad \frac{1}{\lambda_1}\vec{v}_1 = \vec{w}_1$$

$$\frac{1}{\lambda_2}\vec{v}_2 \in X \qquad \frac{1}{\lambda_2}\vec{v}_2 = \vec{w}_2$$

$$\vec{v}_1 + \vec{v}_2 = \lambda_1 \vec{w}_1 + \lambda_2 \vec{w}_2$$

$$= (\lambda_1 + \lambda_2) \cdot \underbrace{\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\vec{w}_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}\vec{w}_2 \right)}_{\in X}$$

by convexity

Definition

Minkowski's successive minima $\lambda_1, \lambda_2, \ldots, \lambda_n$

are:

$$\lambda_k = \inf\{\lambda_{>0} \mid \lambda X \text{ contains } \geq k \text{ linearly independent vectors from } \mathbb{Z}^n\}$$

In particular

$$\lambda_1 = \min\{|\vec{v}|_X \mid \vec{v} \in \mathbb{Z}^n \setminus \{0\}\}$$

Clearly $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$

Minkowski Lemma (1st Theorem)

$$\lambda_1^n \cdot \mathrm{Vol}(X) \leq 2^n$$

We had:

if $vol(X) > 2^n$

then $\lambda_1 \leq 1$

$$vol(\underset{\underset{0}{\vee}}{\lambda_1} X) = \lambda_1^n \cdot vol(X)$$

Minkowski 2nd Theorem

$$\frac{2^n}{n!} \leq \lambda_1 \lambda_2 \cdots \lambda_n vol(X) \leq 2^n$$