Some AG basics

Space (set)

Ring of functions

$x \in X$

$f \longmapsto f(x)$

function

$x \in X \rightsquigarrow I \subset R$

$$I = \{f \in R \mid f(x) = 0\}$$

Affine scheme

R comm. ring with 1

$\mathrm{Spec}(R) = \{ p \subset R \mid p \text{ prime} \}$

$m\mathrm{Spec}(R) = \{ m \subset R \mid m \text{ maximal} \}$

Notes

1) $p \subset R$ is prime

$\Updownarrow$

R/p is a domain

2) $m \subset R$ is maximal

$\Updownarrow$

R/m is a field.

maximal $\Rightarrow$ prime

$$R_1 \xrightarrow{f} R_2$$

$$\cup$$

$$P \quad \text{prime}$$

$f^{-1}(p)$ is also prime

$ab \in f^{-1}(p)$

$\Updownarrow$

$f(ab) \in P$

$\shortparallel$

$f(a) \cdot f(b)$

p is prime $\Rightarrow \begin{cases} f(a) \in P \\ f(b) \in P \end{cases}$

So $a \in f^{-1}(p)$ or $b \in f^{-1}(p)$

Ex. $R = \mathbb{C}[x]$

Maximal ideals

$(x-a) \quad a \in \mathbb{C}$

Prime ideals: maximal and (0)

$$mSpec(R) = \{ m \subset R \mid m \text{ maximal} \}$$

Zariski topology

is generated by

$$U_f = \{ m \mid m \not\ni f \}$$

$$f \in R$$

$R = \mathbb{C}[x]$

$\operatorname{mSpec}(R) \longleftrightarrow \mathbb{C}$

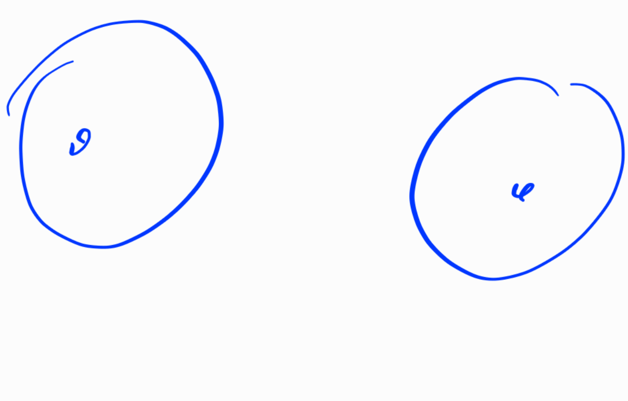
$(x-a) \qquad\qquad a$

$f = f(x)$

$U_f \longleftrightarrow \mathbb{C} \setminus \{\text{roots of } f\}$

Open sets: $\emptyset$,

$\mathbb{C} \setminus A \qquad |A| < \infty$



$U \subset X$

$$\rightsquigarrow$$

$F(U)$ ring

$\mathcal{O}_X(U)$

$R = \mathbb{C}[x]$ $\quad X = \mathbb{C}$

$U = \mathbb{C} \setminus \{2\}$ $\quad$ " $\mathrm{mspec}(\mathbb{C}[x])$

$$f(x) = \frac{x^3 - 2x + 5}{(x-2)^4}$$

$$\mathcal{O}_X(U) = \left\{ \frac{p(x)}{(x-2)^k} \right\}$$

Localization

R comm, with 1

S multiplicative system:

1) $1 \in S$
2) $\forall s_1, s_2 \in S \quad s_1 s_2 \in S$

$$R_S = \left\{ \frac{a}{s} \;\middle|\; a \in R, s \in S \right\}$$

$\frac{a}{s}$ is an equivalence class of pairs:

$$\frac{a_1}{s_1} = \frac{a_2}{s_2}$$

$\Uparrow$

$\Downarrow$

$\exists s_3 \in S$

$s_3(s_2 a_1 - s_1 a_2) = 0$

Ex. $m \subset R$ maximal ideal (prime)

$S = R \setminus m$ mult. system

R_S — ?

$R \longrightarrow R_S$

$a \longmapsto \frac{a}{1}$

$R = \mathbb{C}[x]$ $S = R \setminus m$

$m = (x-2)$

$S = \{ f \in \mathbb{C}[x] \mid f(2) \neq 0 \}$

It is a domain

$R \subset R_S \subset \underbrace{Q(R)}_{\text{field of } R}$

$$R_S = \left\{ \frac{p(x)}{q(x)} \,\middle|\, q(2) \neq 0 \right\}$$

This is a "local ring"

at 2.

It is a ring of functions defined on some neighborhood of 2.

Ex, $f \neq 0$ [Think of $f = x-2$ $\quad U_f = \mathbb{C} \setminus \{2\}$

$$\{1, f, f^2, \ldots\} = S$$

$$R_S = \left\{ \frac{a}{f^k} \right\} \quad f \text{ is a unit in } R_S$$

$$f \cdot \frac{1}{f} = 1$$

This is the ring of functions on U_f

$$U_f = \{ m \in \operatorname{Spec}(R) \mid m \not\ni f \}$$

Maximal ideals in R_S have the form

$$\left\{ \frac{a}{s} \,\middle|\, a \in m \right\}$$

m max in R

$m \not\ni f$

Nullstellensatz (Hilbert)

<u>Thm.</u> Every maximal ideal in $\mathbb{C}[X_1, \dots, X_n]$ is $(X_1 - a_1, X_2 - a_2, \dots X_n - a_n)$ for some $(a_1, \dots, a_n) \in \mathbb{C}^n$

More usual form:

Suppose $(f_1, \dots, f_k)$ is an ideal in $\mathbb{C}[x_1 \dots x_n]$

s.t. $\forall (a_1 \dots a_n) \in \mathbb{C}^n$

$\exists i \in \{1 \dots k\}$ $f_i(a_1 \dots a_n) \neq 0$

Then $(f_1, \dots, f_k) = \mathbb{C}[x_1 \dots x_n]$

$(\exists g_1, \dots, g_k \quad f_1 g_1 + \dots + f_k g_k = 1)$

(poly)

$(f_1 \dots f_k) \subset M_{(a_1 \dots a_n)}$

This means: $\forall i = 1 \dots k$

$$f_i(a_1 \dots a_n) = 0.$$

Idea of the proof

$$\mathbb{C}[x_1, \dots x_n] \mapsto \underbrace{\mathbb{C}[x_1 \dots x_n]/m}_{\text{a field} = K}$$

$$\cup$$

$$\mathbb{C}$$

(constants)

$$\mathbb{C} \hookrightarrow K$$

K is generated over $\mathbb{C}$ as algebra by f. many elements $\bar{x}_1, \dots \bar{x}_n$

Every element of K is a polynomial in $\bar{x}_1, \dots \bar{x}_n$ with coeff. in $\mathbb{C}$.

Theorem (Zariski)

Suppose $k \subset K$, s.t. K is fin. gen. over k as algebra. Then K is an algebraic extension of k.

Idea:

$$K = k(x) = \left\{ \frac{p(x)}{q(x)} \,\middle|\, p, q \in k[x] \right\}$$

$$\underline{f_1} \ldots \underline{f_m} \qquad \frac{1}{f}$$