Closed subgroups of $\mathbb{R}^n$

$$\left\{ \sum_{i=1}^{k} a_i \vec{v}_i + \sum_{i=k+1}^{\ell} m_i \vec{v}_i \,\middle|\, \begin{matrix} a_i \in \mathbb{R} \\ m_i \in \mathbb{Z} \end{matrix} \right\}$$

$\{\vec{v}_1, \dots, \vec{v}_\ell\}$ linearly independent.

<u>Theorem.</u> Closed subgroups of $\mathbb{R}^n$ that contain $\mathbb{Z}^n$ are $\mathbb{Z}^n + W$ where $W \subseteq \mathbb{R}^n$ is a subspace defined by rational relations:

$$\dim_{\mathbb{Q}} W = \dim_{\mathbb{R}} W$$

<u>Proof.</u> ① Every such $\mathbb{Z}^n + W$ will work, and

W will be the connected component of $\{\vec{0}\}$

Consider a $\mathbb{Q}$-basis of W consisting of vectors in $\mathbb{Z}^n$.

We can extend this to a basis of $\mathbb{R}^n$ by adding vectors from $\mathbb{Z}^n$.

We get

$$\underbrace{\vec{v}_1, \ldots, \vec{v}_k}_{\text{span } W}, \underbrace{\vec{v}_{k+1}, \ldots, \vec{v}_n}_{\text{all in } \mathbb{Z}^n}.$$

$\mathbb{Z}^n / \{\sum m_i \vec{v}_i \mid m_i \in \mathbb{Z}\}$ is a finite abelian group.

$\forall i \in \{k+1, \dots, n\}$
the set of i-th coordinates
of points in G in
the basis $\vec{v}_1, \dots, \vec{v}_n$
is discrete.

(2) G W is connected
component of G,

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^n / W$$

$f(G)$ discrete in $\mathbb{R}^n / W$

$G \supseteq \mathbb{Z}^n$
Look at the images
of $\mathbb{Z}^n$ under f.
This is a lattice in $\mathbb{R}^n / W$

$$\cong \mathbb{R}^{n-k}$$

$k = \dim_{\mathbb{R}} W$.

We can take a basis in $\mathbb{R}^n/W$ consisting of points in $f(\mathbb{Z}^n)$

It will have $n-k$ vectors.

$$f(\mathbb{Z}^n) \cong \mathbb{Z}^n / (W \cap \mathbb{Z}^n)$$

$$\operatorname{Rank}(f(\mathbb{Z}^n)) = n-k$$

So $\operatorname{rank}(W \cap \mathbb{Z}^n) = k$,

W is defined over $\mathbb{Q}$. $\square$

Kronecker Theorem

X centrally symmetric convex closed bounded set in $\mathbb{R}^n$

$$|\vec{v}| = \inf\{\lambda \mid \lambda X \ni \vec{v}\}$$

$\lambda \in \mathbb{R}_{\geq 0}$

$\mathbb{R}^n$

$$X = \{\vec{v} \in \mathbb{R}^n \mid |\vec{v}| \leq 1\}$$

<u>Def.</u> $\lambda_1, \ldots \lambda_n$

$$\lambda_k = \inf\{\lambda \mid \lambda X \cap \mathbb{Z}^n \text{ has } k \text{ linearly independent vectors}\}$$

<u>Minkowski's lemma</u>

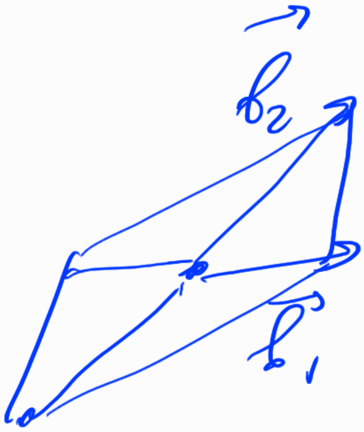
$$\lambda_1^n \operatorname{vol}(X) \leq 2^n$$

<u>Minkowski's 2nd theorem:</u>

$$\frac{2^n}{n!} \leq \lambda_1 \cdots \lambda_n \operatorname{vol}(X) \leq 2^n$$

Proof. We can find a basis of $\mathbb{R}^n$:
$\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ s.t.
$|\vec{b}_i| = \lambda_i$, $\vec{b}_i \in \mathbb{Z}^n$,

Consider
$$\tilde{X} = \operatorname{conv}\left(\pm \frac{1}{\lambda_i}\vec{b}_i\right) \subseteq X$$
$$\underset{\overset{\wedge}{\operatorname{vol}(X)}}{\operatorname{vol}(\tilde{X})} = \left(\frac{1}{\prod \lambda_i}\right) \cdot \underset{\overset{\parallel}{\frac{1}{\prod\lambda_i} \cdot \frac{2^n}{n!}}}{\operatorname{vol}\operatorname{conv}(\pm \vec{b}_i)}$$



$$\left(\prod \lambda_i\right) \operatorname{vol}(X) \geq \frac{2^n}{n!}$$

Incorrect proof of the other inequality:

The set of all vectors in $\mathbb{R}^n$ with coordinates in $[-1,1]^n$ in the basis $\{\frac{v_i}{\lambda_i}\}$.

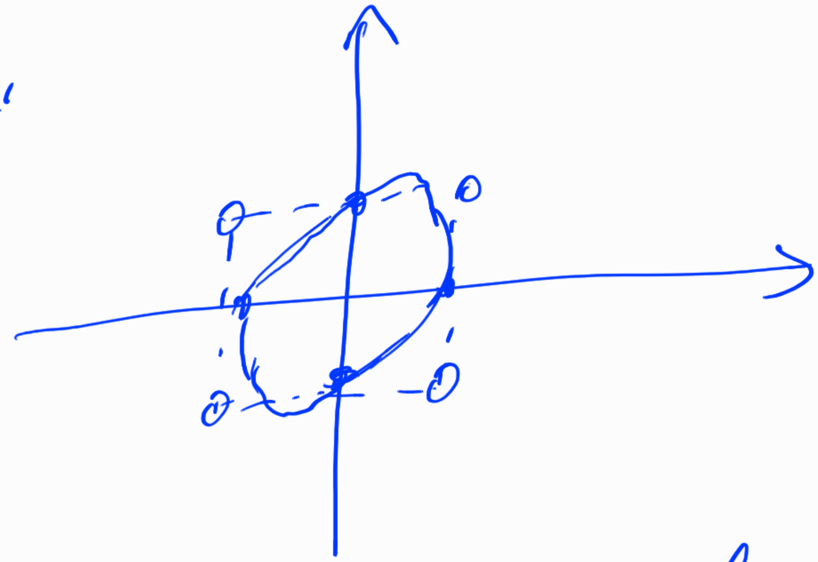
$$\mathrm{vol}(X) \leq \frac{1}{\prod \lambda_i} \cdot 2^n$$

$$\left(\prod \lambda_i\right) \mathrm{vol}(X) \leq 2^n$$

Mistake: X does not have to be inside that set.

Ex.

$\lambda_1 = \lambda_2 = 1$



Now correct proof.

$\forall i \in \{1, \dots, n\}$

$$c_i(\vec{x}) = \text{centroid of } \underline{\left(\vec{x} + \operatorname{span}(\vec{b}_1 \dots \vec{b}_{i-1})\right)} \cap X$$

$c_i(\vec{x}) \in X$

$$c_1(\vec{x}) = \vec{x}$$

$$c_i(\vec{x}) \in \vec{x} + \operatorname{span}(\vec{b}_1, \dots, \vec{b}_{i-1})$$

$c_i(\vec{x})$ only depends on $x_i, \dots, x_n$ where $\vec{x} = \sum_{i=1}^{n} x_i \vec{b}_i$

$$F(\vec{x}) = \left(\sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})\, c_i(\vec{x})\right)/2$$

$\cap$

X

$$F(X) \subseteq \mathbb{R}^n$$

we will show:

1) $\operatorname{vol} F(X) = \left[\left(\prod \lambda_i\right)/2^n\right] \operatorname{vol}(X)$

2) If $F(y) - F(y') \in \mathbb{Z}^n$

$\cap \qquad \cap$

$X \qquad X$

then $F(y) = F(y')$

(because $y = y'$)

This will be enough:

$$\frac{\lambda_1 \cdots \lambda_n}{2^n} \operatorname{vol}(X) \leq 1$$

To prove (1) we will show that

$$F\left(\sum x_i \vec{v_i}\right) = \sum z_i \vec{v_i}$$

where for each i

$$z_i = \frac{\lambda_i}{2} x_i + f_i(x_{i+1}, \dots, x_n)$$

continuous