$X \subseteq \mathbb{R}^n$

closed, convex, compact, centrally symmetric

$vol(X) > 0$

Then $\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n \, vol(X) \leq 2^n$

Proof: $\exists$ a basis of $\mathbb{R}^n$

$\{\vec{b}_1, \ldots, \vec{b}_n\}$ s.t.

$\vec{b}_i \in \mathbb{Z}^n$

and $|\vec{b}_i| = \lambda_i$

$$\left(\begin{array}{l} \vec{b}_i \in \lambda_i X \\ \vec{b}_i \notin \mu X \quad \forall \mu \in (0, \lambda_i) \end{array} \right)$$

$\forall i \in \{1, \ldots, n\}$ we

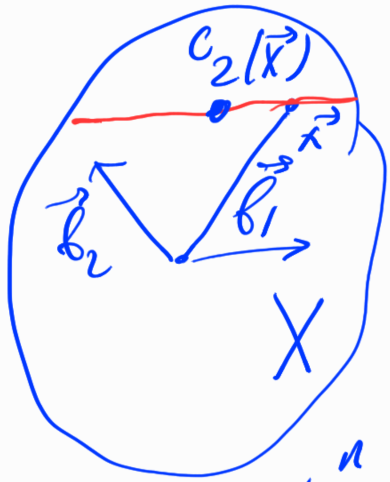
consider $C_i(\vec{x})$ is

$\cap$

X

the centroid of

$X \cap (\vec{x} + span(\vec{b}_1, \ldots, \vec{b}_{i-1}))$



$d=2$

$i=1$

$$F(\vec{x}) = \left(\sum_{i=1}^{n} (\lambda_i - \lambda_{i-1}) c_i(\vec{x})\right) / 2$$

$\vec{x} \in X$

We will prove:

1) $\mathrm{vol}(F(X)) = \frac{\prod \lambda_i}{2^n} \mathrm{vol}(X)$

2) If $F(\vec{y}) = F(\vec{y}') + \vec{c}$, $\vec{c} \in \mathbb{Z}^n$

for $\vec{y}, \vec{y}' \in X$

then $\vec{y} = \vec{y}'$, so $\vec{c} = 0$.

<u>Proof of 1</u>

We will show:

If $\vec{x} = \sum x_i \vec{b}_i$

and $F(\vec{x}) = \sum z_i \vec{b}_i$

then $\forall i \quad z_i = \frac{\lambda_i}{2} x_i + f_i(x_{i+1}, \ldots x_n)$

Let's analyze how i-th coordinate of $C_k(\vec{x})$ is related to x_i

1) If $k \le i$, then the i-th coordinate of $C_k(\vec{x})$ is x_i.

2) If $k > i$, then the i-th coordinate of $C_k(\vec{x})$ only depends on $x_k, x_{k+1}, \dots, x_n$ so in particular it only depends on $x_{i+1}, \dots, x_n$

$$F(\vec{x}) = \left(\sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) C_k(\vec{x})\right)/2$$

$$= \left(\sum_{k=1}^{i} (\lambda_k - \lambda_{k-1}) C_k(\vec{x})\right)/2 + \left(\sum_{k=i+1}^{n} (\lambda_k - \lambda_{k-1}) C_k(\vec{x})\right)/2$$

The i-th coordinate z_i of $F(\vec{x})$ is:

$$\frac{1}{2}\sum_{k=1}^{i}(\lambda_k-\lambda_{k-1})x_i+f_i(x_{i+1},\dots,x_n)$$

$$\parallel$$

$$\frac{\lambda_i}{2}x_i+f_i(x_{i+1},\dots,x_n)$$

<u>Proof of 2</u>

Suppose $F(y)=F(y')+c$, $c\in\mathbb{Z}^n$

$y, y'\in X$

$y=\sum x_i\vec{b}_i$, $y'=\sum x_i'\vec{b}_i$

Suppose $K=\max(i \mid x_i\neq x_i')$

Then

$$F(y)-F(y')$$

$$\parallel$$

$$\frac{\lambda_k}{2}(x_i-x_i')\vec{b}_k+\sum_{i=1}^{k-1}?\,\vec{b}_i$$

Suppose

$$F(\vec{y})-F(\vec{y}')=\vec{c}\in\mathbb{Z}^n$$

$$F(\vec{y}) - F(\vec{y}\,')$$

$$\frac{F}{2} \overset{\text{"}}{\sum_{i=1}^{K}} (\lambda_i - \lambda_{i-1}) \left[c_i(\vec{y}) - c_i(\vec{y}\,') \right].$$

$$\left(\sum_{i=1}^{K} (\lambda_i - \lambda_{i-1}) = \lambda_K \right.$$

$$\lambda_K \cdot \frac{\sum_{i=1}^{K} \frac{\lambda_i - \lambda_{i-1}}{\lambda_K} c_i(\vec{y}) - \sum_{i=1}^{K} \frac{\lambda_i - \lambda_{i-1}}{\lambda_K} c_i(\vec{y}\,')}{2}$$

$$\wedge$$

$$\lambda_K X$$

In fact, in the <u>interior</u> of X

So we have a point in $\mathbb{Z}^n$ in the interior of $\lambda_k X$, not in span $(\vec{b}_1, \dots, \vec{b}_{k-1})$

Contradicts definition of λ_k $\square$

Duality:

Def. Suppose $X \subseteq \mathbb{R}^n$ is full-dimensional convex closed centrally symmetric.

Then $pol(X) = \{ f \in (\mathbb{R}^n)^* \mid \forall \vec{x} \in X \ \ f(\vec{x}) \leq 1 \}$

Theorem. $pol(pol(X)) = X$

$X, \ \mathbb{Z}^n \rightsquigarrow \lambda_1, \ldots, \lambda_n$

$X^* = pol(X), (\mathbb{Z}^n)^* \rightsquigarrow \lambda_1^* \ldots \lambda_n^*$

Thm. 1 $\forall i \in \{1, \ldots, n\}$

$$\lambda_i \cdot \lambda_{n+1-i}^* \geq 1$$

Thm 2. $\forall i \in \{1, \ldots, n\}$

$$\lambda_i \cdot \lambda_{n+1-i}^* \leq (n!)^2$$

(Transference Inequality)

Proof of Thm 1.

$\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ for $X, \mathbb{Z}^n$

$\vec{b}_1^*, \ldots, \vec{b}_n^*$ for $X^*, (\mathbb{Z}^n)^*$,

(Not the dual basis)

$\vec{b}_i \in \lambda_i X \quad \vec{b}_j \in \lambda_j^* X^*$

$$|\vec{b}_j^*(\vec{b}_i)| \le \lambda_i \lambda_j^*$$

Suppose $\lambda_i \cdot \lambda_{n+1-i}^* < 1$

Then $\forall i' \le i, \quad \forall j' \le n+1-i$

$$|\vec{b}_{j'}^*(\vec{b}_{i'})| \le \lambda_{i'} \cdot \lambda_{j'}^* \le \lambda_i \lambda_j^* < 1$$

$\mathbb{Z}$

So $\quad \vec{b}_{j'}^*(\vec{b}_{i'}) = 0$

$\mathrm{Span}(\vec{b}_1 \ldots \vec{b}_i)$ $\quad i$

is perpendicular to

$\mathrm{Span}(\vec{b}_1^*, \ldots, \vec{b}_{n+1-i}^*)$ $\quad n+1-i$

$i + (n+1-i) = n+1 > n$, impossible $\square$

Proof of Thm 2.

$$\lambda_i \lambda^*_{n+1-i}$$

$\geq$ by Thm 1

$$(\lambda_1 \lambda_n^*)\cdot(\lambda_2 \lambda_{n-1}^*)\cdots(\lambda_n \lambda_1^*)$$

$$=$$

$$\left(\prod \lambda_i\right)\cdot \prod(\lambda_i^*)$$

$$\geq$$

$$\frac{2^n}{vol(X)}\cdot\frac{2^n}{vol(X^*)} = \frac{4^n}{vol(X)\cdot vol(X^*)}$$

We will prove that

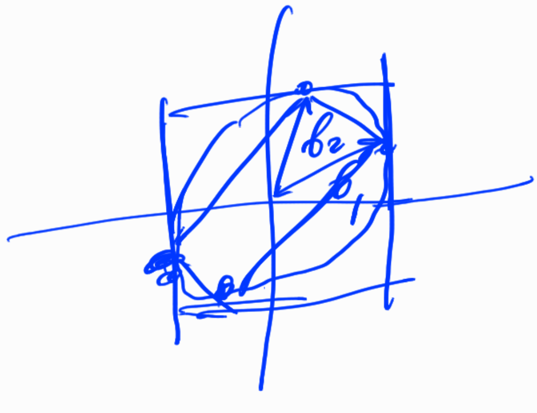
$$vol(X)\cdot vol(X^*) \geq \left(\frac{2^n}{n!}\right)^2$$

$\forall i$ consider

$$\max\left(x_i \mid \vec{x}=(x_1,\dots x_n)\in X\right)=l_i$$

It is achieved at some

$$\vec{b}_i \in \vec{X}$$



$$\vec{X}^* \supseteq \left(\prod_{i=1}^{n} [-c_i, c_i] \right)^*$$

$\{\vec{b}_1, \ldots \vec{b}_n\}$. a basis

to be continued ...