$X, \mathbb{Z}^n \qquad \lambda_1 \dots \lambda_n$

$X^*, (\mathbb{Z}^n)^* \qquad \lambda_1^* \dots \lambda_n^*$

Thm. $\forall i = 1, 2, \dots, n$

$$1 \le \lambda_i \lambda_{n+1-i}^* \le (n!)^2$$

Proof. ① $1 \le \lambda_i \lambda_{n+1-i}^*$

② $\lambda_i \lambda_{n+1-i}^* \le \left(\prod \lambda_i\right) \prod \left(\lambda_i^*\right)$

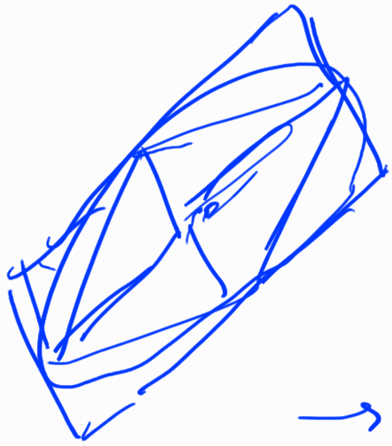
$$\le \left(\frac{2^n}{\mathrm{vol}(X)}\right)\left(\frac{2^n}{\mathrm{vol}(X^*)}\right)$$

$$" \frac{4^n}{\mathrm{vol}(X) \cdot \mathrm{vol}(X^*)}$$

We will show:

$$\mathrm{vol}(X)\cdot\mathrm{vol}(X^*)\geq\left(\frac{2^n}{n!}\right)^2$$

To prove this:



We take $\vec{v}_1, \dots \vec{v}_n$ in X

s.t. $|\det(\vec{v}_1 \dots \vec{v}_n)|$

is maximal possible.

$\mathrm{conv}(\pm\vec{v}_i)$ generalized

octahedron has

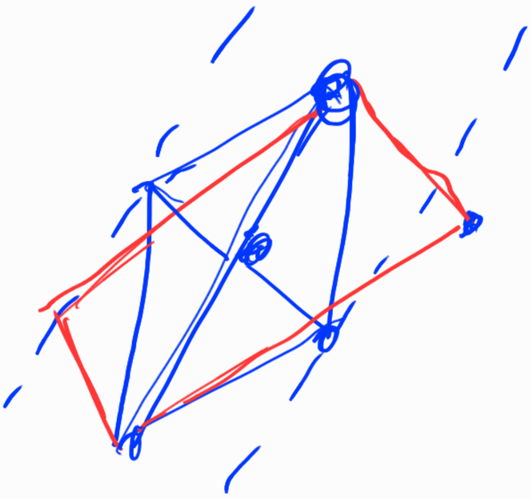
max possible volume out

of all generalized octahedra

in X.

$$\mathrm{vol}\left(\mathrm{conv}(\pm\vec{v_i})\right) = \frac{2^n}{n!}\left|\det(\vec{v_1}\ldots\vec{v_n})\right|$$

<u>Claim.</u> All points of X have coordinates in $[-1,1]^n$ in the basis $\{\vec{v_1},\ldots,\vec{v_n}\}$.



If $\vec{x} = \sum x_i \vec{v_i}$ with $x_n > 1$

$\cap$

X

then $\det|\vec{v_1}\ldots\vec{v}_{n-1}, \vec{x}|$

$= \det|\vec{v_1}\ldots\vec{v}_{n-1}, x_n\vec{v_n}|$

$= x_n \cdot \det|\vec{v_1} \cdots \vec{v_{n-1}}, \vec{v_n}|$

So if $|x_n| > 1$, contradiction.

$$\operatorname{conv}(\pm \vec{v_i}) \subseteq X \subseteq \{\sum a_i \vec{v_i} \mid |a_i| \leq 1\}$$

$\swarrow$

$$X^* \supseteq \{\sum a_i \vec{v_i} \mid |a_i| \leq 1\}^*$$

We can consider the dual basis $\vec{v_i}^*$ in $(\mathbb{R}^n)^*$:

$$\vec{v_i}^*(\vec{v_j}) = \delta_{ij}$$

$$\forall i \quad \pm \vec{v_i}^* \in X^*$$

(In fact, $\{\sum a_i \vec{v_i} \mid |a_i| \leq 1\}^*$

$$\parallel$$

$$\operatorname{conv}\{\pm \vec{v_i}^*\}$$

$$\det(\vec{v}_1 \dots \vec{v}_n) \cdot \det(\vec{v}_1^* \dots \vec{v}_n^*) = 1$$

$$\begin{cases} vol(X) \geq \dfrac{2^n}{n!} \det(\vec{v}_1 \dots \vec{v}_n) \\ vol(X^*) \geq \dfrac{2^n}{n!} \det(\vec{v}_1^* \dots \vec{v}_n^*) \end{cases}$$

$$vol(X)\, vol(X^*) \geq \frac{4^n}{(n!)^2} \qquad \square$$

Minkowski sum

A, B two sets in $\mathbb{R}^n$

$$A + B = \{ \vec{z} \mid \exists \vec{x} \in A\ \exists \vec{y} \in B\ \ \vec{z} = \vec{x} + \vec{y} \}$$

If A, B are convex, then A+B is convex:

$$\begin{cases} \vec{x}+\vec{y} \in A+B \\ \vec{x}'+\vec{y}' \in A+B \end{cases}$$

$0 \le \alpha \le 1$

Need: $\alpha(\vec{x}+\vec{y})+(1-\alpha)(\vec{x}'+\vec{y}')$

$$= \underbrace{(\alpha\vec{x}+(1-\alpha)\vec{x}')}_{\in A}+\underbrace{(\alpha\vec{y}+(1-\alpha)\vec{y}')}_{\in B}$$

$\in A+B$.

So we get a commutative semigroup of all convex, closed sets.

Thm. (Minkowski) This is a cancellation semigroup.

Proof. Suppose $f \in (\mathbb{R}^n)^*$ $f: \mathbb{R}^n \to \mathbb{R}$

Suppose A is convex, closed compact set in $\mathbb{R}^n$. Then $f(A)$ is a convex closed set in $\mathbb{R}$; bounded:

$$[a_1, a_2]$$

Same for B:

$$f(B) = [b_1, b_2]$$

,

claim. $f(A+B)$

$\parallel$

$f(A)+f(B)=[a_1+a_2, b_1+b_2]$

$\sup(f(\vec{x}+\vec{y}) \mid \vec{x}\in A, \vec{y}\in B)$

$\parallel$

$\sup(f(\vec{x}) \mid \vec{x}\in A)+\sup f(\vec{y} \mid \vec{y}\in B)$

Suppose $A+C=B+C$

$\forall f\in(\mathbb{R}^n)^*$

$f(A)=[a_1, a_2]$, $f(B)=[b_1, b_2]$

$f(C)=[c_1, c_2]$

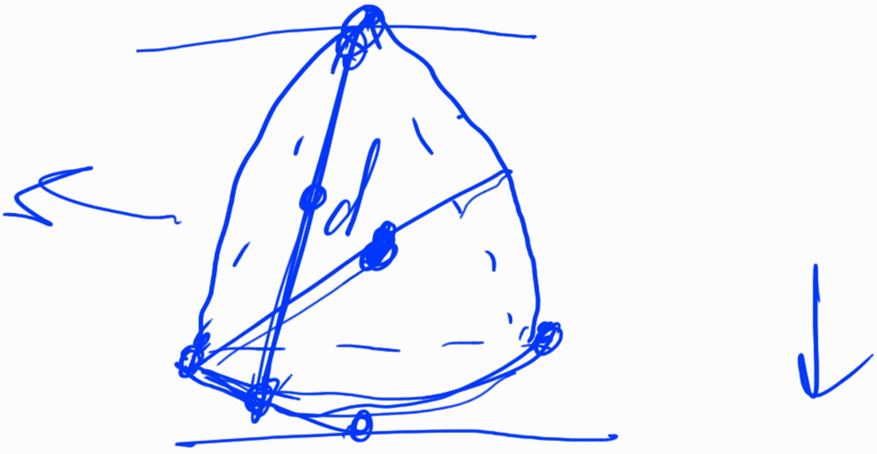
$$\begin{cases} a_1+c_1=b_1+c_1 \\ a_2+c_2=b_2+c_2 \end{cases} \Rightarrow \begin{cases} a_1=b_1 \\ a_2=b_2 \end{cases}$$

Every convex set A

A, B convex $\Rightarrow A+B$.

__Remark.__ A is determined uniquely by $\{f(A) \mid f \in (\mathbb{R}^n)^*\}$ but not by $|f(A)|$.

__Def.__ $A \subset \mathbb{R}^n$ convex compact. It is called a figure of constant width if all projections have the same length



Lawrence 1988

Lemma:

$U \subseteq \mathbb{R}^n$ convex, $p \in U$, $H \subset \mathbb{R}^n$ subgroup

s.t. $H \cap \left(\frac{1}{n}(U-p)\right)$

has a basis for $\mathbb{R}^n$.

Then $H + U = \mathbb{R}^n$

<u>Proof.</u> Suppose

$\vec{b}_1 \dots \vec{b}_n$ is the basis

of $\mathbb{R}^n$ $\quad \forall \vec{b}_i \in H \cap \left(\frac{1}{n}(U-P)\right)$

Suppose $\quad \vec{x} = \sum x_i \vec{b}_i$

$$\vec{x} \in \underset{\substack{\cap \\ H}}{h} + \sum \{x_i\} \vec{b}_i$$

$$0 \le \{x_i\} \le 1$$

$$\forall i \quad n\vec{b}_i \in U-P$$

$$0 < \sum \{x_i\} \le n$$

$$\underset{\substack{\cap \\ U-P}}{\sum \{x_i\} \vec{b}_i} = \left(\sum \frac{\{x_i\}}{\sum \{x_i\}} \underset{\substack{\cap \\ U-P}}{\boxed{\left(\sum \{x_i\}\right) \vec{b}_i}} \right)$$

U $\qquad\qquad$ $U-P$

$$H + (U-P) = \mathbb{R}^n$$

So $H + U = \mathbb{R}^n$. $\square$

<u>Lemma 2.</u>

K full-dimensional, convex, compact, $K = -K$.
$H \subseteq \mathbb{R}^n$ closed s.t.
$H \cap K$ has no basis for $\mathbb{R}^n$.
Then $H^* \cap ((n!)^2 \cdot K^*)$ has a non-zero element.

Proof. Suppose first that M is a lattice. Then this really means: λ_n for K, M is greater than 1. And if $M^* \cap ((n!)^2 K^*)$ has no non-zero element, this implies that λ_1^* for K^*, M^* is $> (n!)^2$.

But $\lambda_n \cdot \lambda_1^* \leq (n!)^2$, a contradiction.