Lemma 2.

K full-dim compact convex, $K = -K$ in $\mathbb{R}^n$

H closed subgroup in $\mathbb{R}^n$

If $K \cap H$ doesn't contain a basis of $\mathbb{R}^n$

then $(n!)^2 K^* \cap H^*$ has a non-zero element.

Proof. If H is a lattice:

$K, H \quad \lambda_1 \ldots \lambda_n$

$K^*, H^* \quad \lambda_1^* \ldots \lambda_n^*$

$$\lambda_n \cdot \lambda_1^* \leq (n!)^2$$

$$H^* = \mathrm{Hom}(H, \mathbb{Z})$$

$$K^* = \mathrm{pol}(K) = \{ f : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } \forall \vec{x} \in K \ f(\vec{x}) \leq 1 \}$$

General case:

by a change of basis:

$$H = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{array}{l} x_1, \dots, x_a \in \mathbb{R} \\ x_{a+1}, \dots, x_{a+b} \in \mathbb{Z} \\ x_{a+b+1}, \dots, x_n = 0 \end{array} \right\}$$

$a, b \geq 0$

Take m large enough

Consider

$$H_m = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{array}{l} x_1, \dots, x_a \in \frac{1}{m}\mathbb{Z} \\ x_{a+1}, \dots x_{a+b} \in \mathbb{Z} \\ x_{a+b+1}, \dots, x_n \in m\mathbb{Z} \end{array} \right\}$$

$$H_m^* = \left\{ [x_1, \dots x_n] \;\middle|\; \begin{array}{l} x_1, \dots, x_a \in m\mathbb{Z} \\ x_{a+1}, \dots, x_{a+b} \in \mathbb{Z} \\ x_{a+b+1}, \dots x_n \in \frac{1}{m}\mathbb{Z} \end{array} \right\}$$

K full-dimensional

K^* bounded.

$H_m^* \cap K^* \subseteq H^* \cap K^*$

for large enough m

$H_m \cap K$ has no bases

of $\mathbb{R}^n$. So $\exists$

$\vec{v} \in H_m^* \cap K^* \subseteq H^* \cap K^*$ $\square$

$\neq \vec{0}$

A little more on

duality for closed

subgroups of $\mathbb{R}^n$.

$H \subseteq \mathbb{R}^n$

closed

<u>Def.</u> $H^* = \{ f \in (\mathbb{R}^n)^* \mid f(H) \subseteq \mathbb{Z} \}$

H^* is a closed subgroup of $(\mathbb{R}^n)^*$.

<u>True:</u> 1) $(H^*)^* \cong H$.

$$H = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{array}{ll} 1 \le i \le a & x_i \in \mathbb{R} \\ a+1 \le i \le a+b & x_i \in \mathbb{Z} \\ a+b+1 \le i \le n & x_i = 0 \end{array} \right\}$$

$$H^* = \left\{ [x_1 \dots x_n] \middle| \begin{array}{ll} 1 \le i \le a & x_i = 0 \\ a+1 \le i \le a+b & x_i \in \mathbb{Z} \\ a+b+1 \le i \le n & x_i \in \mathbb{R} \end{array} \right\}$$

2) $(H_1 + H_2)^* = H_1^* \cap H_2^*$

$(H_1 \cap H_2)^* = H_1^* + H_2^*$

Clear:

If $H \subsetneq G$

then $H^* \supsetneq G^*$

(if $H^* = G^*$, $\underset{H}{\underset{\|}{(H^*)^*}} = \underset{G}{\underset{\|}{(G^*)^*}}$)

An element f of $H^* \setminus G^*$ is a linear functional that takes integer values on H, but not on G.

Lemma 3. Suppose G is a closed subgroup of $\mathbb{R}^n$. Suppose U is a relatively open nonempty subset of G. Then $\exists$ a bounded set $X \subseteq (\mathbb{R}^n)^*$ s.t.

$\forall$ closed subgroup H of G s.t. $H+U \neq G$

$$(H^* \setminus G^*) \cap X \neq \emptyset$$

<u>Proof.</u> By choosing the basis, we can assume that

$$G = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{array}{ll} 1 \le i \le a & x_i \in \mathbb{R} \\ a+1 \le i \le a+b & x_i \in \mathbb{Z} \\ a+b+1 \le i \le n & x_i = 0 \end{array} \right\}$$

$$\mathbb{R}^n = A \oplus B \oplus C$$

$$A = \begin{bmatrix} x_1 \\ \vdots \\ x_a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{a+1} \\ x_{a+b} \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{a+b+1} \\ \vdots \\ x_n \end{bmatrix}$$

We identify $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ using this basis.

$\alpha : \mathbb{R}^n \to A$ projections

$\beta : \mathbb{R}^n \to B$

$\gamma : \mathbb{R}^n \to C$

$P \subset B$

$$P = \left\{ \begin{pmatrix} 0 \\ \vdots \\ x_{a+1} \\ \vdots \\ x_{a+b} \\ \vdots \\ 0 \end{pmatrix} \,\middle|\, \forall i \;\; 0 \le x_i \le 1 \right\}$$

$\Big($Note: If $a=0$, $b=n$, $G=\mathbb{Z}^n$

$G^* = \mathbb{Z}^n$; $\forall H \subsetneq G$

$\exists\, h^* \in H^* \setminus G^*$ in $P\Big)$

Take a point $p \in U$

Consider $W = \{ \vec{x} \in A \mid \sum x_i^2 \le 1 \}$

$(W^* = W$ in $A)$

Take $\varepsilon > 0$ s.t.

$$\varepsilon W \subset \frac{U - p}{a}$$

$$X = \frac{(a!)^2}{\varepsilon} W + P$$

Suppose $H \subsetneq G$

$H + U \neq G$

Easy case $\beta(H) \neq \beta(G)$

$\beta(G) \cong \mathbb{Z}^b \subset B$

$$\beta^{-1}\beta(H) \neq \beta^{-1}\beta(G)$$

$$\begin{array}{ccc} \beta^{-1}\beta(H) & & \beta^{-1}\beta(G) \\ \| & & \| \\ A+C+H & & A+G\cap B+C \\ & & \| \\ & & A+G+C \end{array}$$

$$\underbrace{A+C}+H \neq \underbrace{A+C}+G$$

Taking duals

$$B \cap H^* \neq B \cap G^*$$

$$\supset$$

$$x = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{a+1} \\ \vdots \\ x_{a+b} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad B \cap G^* = \left\{ \begin{bmatrix} 0 \\ \vdots \\ \mathbb{Z}^b \\ \vdots \\ 0 \end{bmatrix} \right\}$$

By subtracting from

$$\vec{x} \quad \begin{bmatrix} 0 \\ [x_{a-1}] \\ [x_{a+\beta}] \\ \vdots \\ 0 \end{bmatrix} \in B \cap G^{\alpha}$$

we get an element in P, which is in H, not in G^{α}.

<u>Hard case.</u>

$$\beta(H) = \beta(G)$$

Lemma 1 applied to

$$A \cap H \subsetneq A \cap G$$

$$H + (U - P) \neq G \qquad \begin{aligned} &H + (U - P) \\ &\quad \parallel \\ &(H + U) - P \end{aligned}$$

$$\varepsilon W \subseteq \frac{U - P}{a}$$

$$\varepsilon a W \subseteq U - P$$

$$H + a(\varepsilon W) \neq G$$

$$H \cap A + a(\mathcal{E}W) \neq \underset{\overset{\shortparallel}{A}}{G \cap A}$$

<u>Lemma 1</u>: If

$$H \cap A + a(\mathcal{E}W) \neq G \cap A$$

then $\nexists$ basis in $\underline{H \cap \mathcal{E}W}$

<u>Lemma 2.</u>