Lemma 3.

G is a closed subgroup of $\mathbb{R}^n$, $U \subset G$ is a relatively open set, $p \in U$. Then $\exists$ bounded set $X \subset (\mathbb{R}^n)^*$ s.t. $\forall$ closed $H \subset G$ s.t. $H + U \neq G$ $\exists f \in X \cap (H^* \setminus G^*)$

Proof. $G = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{array}{ll} 1 \leq i \leq a & x_i \in \mathbb{R} \\ a+1 \leq i \leq a+b & x_i \in \mathbb{Z} \\ a+b+1 \leq i \leq n & x_i = 0 \end{array} \right\}$

$\mathbb{R}^n = A \oplus B \oplus C$

$\alpha : \mathbb{R}^n \to A$

$\beta : \mathbb{R}^n \to B$

$\gamma : \mathbb{R}^n \to C$

$(\mathbb{R}^n)^* = \mathbb{R}$

$$\beta(G) = \left\{ \begin{bmatrix} \vdots \\ \mathbb{Z}^b \\ 0 \\ \vdots \end{bmatrix} \right\} \subset B$$

$$P \subset B = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_i \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \;\middle|\; \begin{matrix} \forall i \\ a+1 \leq i \leq a+b+1 \\ 0 \leq x_i \leq 1 \end{matrix} \right\}$$

$W \subset A$ unit ball

$$W = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \;\middle|\; x_1^2 + \dots + x_a^2 \leq 1 \right\}$$

$W = W^*$ (identifying $(\mathbb{R}^a)^*$ with $\mathbb{R}^a$)

Take $\varepsilon > 0$ s.t.

$$\varepsilon W \subseteq \frac{U - P}{a}$$

$$X = \frac{(a!)^2}{\varepsilon} W + P$$

1) Easy case: $\beta(H) \neq \beta(G)$
$\beta(G) = \mathbb{Z}^b \subseteq B$

2) Hard case: $\beta(H) = \beta(G)$

Lemma 1 applied to

$H \cap A, \quad \varepsilon W \qquad \varepsilon W \subset \frac{U-p}{a}$

$\underline{H + U \neq G} \qquad a\varepsilon W \subset U - p$

$\Downarrow$

$H + (U - p) \neq G$

$\Downarrow$

$H + a\varepsilon W \neq G$

$\Downarrow$

$H \cap A + a\varepsilon W \neq G \cap A$

(if $\vec{x} \in G$, $\vec{x} \notin H + a\varepsilon W$

$\exists \vec{x}' \in H$ s.t. $\beta(\vec{x}) = \beta(\vec{x}')$)

$\vec{x} - \vec{x}' \in A$

$\vec{x} - \vec{x}' \notin H + a\varepsilon W$

Lemma 1 applied to

$H \cap A$, $a\varepsilon W$ implies

$H \cap \varepsilon W$ has no

basis for A.

Lemma 2 applied to

$H \cap A \subset A$, εW:

$$\frac{(a!)^2}{\varepsilon} W \cap (H \cap A)^* \ni f$$

$$f \neq \vec{0}$$

$$H \cap A \subset A < \mathbb{R}^n$$

$(H \cap A)^*$ meaning #1 in A^*

meaning #2 in $(\mathbb{R}^n)^*$

f is $A \longrightarrow \mathbb{R}$

s.t. $f(H \cap A) \in \mathbb{Z}$

<u>Red is wrong</u>

Extend f to $\mathbb{R}^n \to \mathbb{R}$

by $f|_B = f|_C = 0$

Then $f \in H^* \setminus G^*$

and $f \notin X$.

<u>Problem</u>

$$H \cap A \neq \alpha(H)$$

<u>Ex.</u> $a=1,\ b=1,\ c=0$

$$G = \mathbb{R} + \mathbb{Z} \subseteq \mathbb{R}^2$$

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \begin{array}{l} x_1, x_2 \in \mathbb{Z} \\ x_1 + x_2 \equiv 0 \bmod 2 \end{array} \right\}$$

$$H \cap A = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \middle| x_1 \in 2\mathbb{Z} \right\}$$

But $\mathcal{L}(M) = \mathbb{Z}$

So $f : \mathbb{R}^2 \to \mathbb{R}$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{2} x_1$$

$$\frac{1}{2} x_1 \in (H \cap A)^*$$

$$\notin M^*$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in H$$

$$(H \cap A)^* = H^* + A^* \quad \leftarrow \text{in } \mathbb{R}^n$$

$$= \underline{H^*} + B + C$$

$f : H \cap A \to \mathbb{R}$

extend it to $\mathbb{R}^n \to \mathbb{R}$

$f \notin H^*$ but

$$f \in H^* + \underline{B} + \underline{C}$$

$\exists\ \tilde{f} : \mathbb{R}^n \to \mathbb{R}$

$\cap$

H^* that extends

$f : A \to \mathbb{R}$ $\frac{(a!)^2}{\varepsilon} w$

s.t.

$$\tilde{f} = \begin{bmatrix} 0 & n \\ 0 & \\ \vdots & \end{bmatrix}$$

By subtracting $\langle x_i \rangle$

for $i \in [a+1, a+b+1]$

we can find

$\tilde{f}$ $\begin{bmatrix} 0 \\ 0 \\ \leftarrow \end{bmatrix}$ $\begin{array}{l} n \ \frac{(a!)^2}{\varepsilon} w \\ n \ P \\ \text{make} \\ \text{this } 0 \end{array}$

$\cap$

H^*

So $\tilde{f} \in X$ $\tilde{f} \in H^*$

$\tilde{f} \notin G^*$ because

$\mathcal{L}(\tilde{f}) \neq \vec{0}$.

$\square$

Main Theorem

Def. $U \subseteq G$ is a full set if $\forall$ closed subgroup $H \leq G$

$H \cap U$ is empty or relatively open

Note: relatively open $\Rightarrow$ full

empty $\Rightarrow$ full

Def. $U \subset G$ is a full set.

$M(G, U)$ is the set of all closed $\mathbb{Z}^n \leq H \leq G$ subgroup s.t. $H \cap U = \emptyset$ maximal under inclusion.

Thm. Suppose G is a closed subgroup of $\mathbb{R}^n$ that contains $\mathbb{Z}^n$ and $U \subset G$ is any full set. Then $M(G,U)$ is finite.

Proof. Suppose G is such that $\exists\, U \subset G$, full, for which $M(G,U)$ is infinite.

Suppose $H \in M(G,U)$

$$G \supset$$

$$U \neq \emptyset \quad (U = \emptyset \Rightarrow M(G,U) = \{G\})$$

$$H + U \neq G.$$

$$H + U \not\ni \vec{0} \in G$$

So by lemma 3

$\exists\, \vec{b} \in X$ ← fixed bounded set

$\vec{b} \in M^* \setminus G^*$ depending on G, U

$b : H \to \mathbb{Z}$

$b : G \not\to \mathbb{Z}$

Consider $G_b \subset G$

$$G_b = \{\vec{x} \in G \mid b(\vec{x}) \in \mathbb{Z}\}$$

$$\underline{H \subseteq G_b} \subsetneq G$$

$$U_b = G_b \cap U$$

$$H \in M(G_b, U_b)$$

$$M(G, U) \subseteq \bigcup_{b \in \vec{X}} M(G_b, U_b)$$

$$(\mathbb{Z}^4)^* \ni \vec{b} \notin G^*$$

$\vec{b} \in H^{\perp} \subseteq (\mathbb{Z}^n)^*$

$H \supseteq \mathbb{Z}^n$

X bounded, $(\mathbb{Z}^n)^*$ discrete

So $\{\vec{b}\}$ is finite.

$M(f, U)$ is infinite

$\Downarrow$

$\exists \vec{b}$ $M(G_b, U_b)$ is infinite.

The set of G s.t. $\exists U$ $|M(f, U)| = \infty$ has no minimal element.

But the set of all closed subgroups of $\mathbb{R}^n$ that contain $\mathbb{Z}^n$

satisfies descending chain condition:

$$G_1 \supseteq G_2 \supseteq \ldots \supseteq G_k \supseteq \ldots$$

$$\exists K \ \forall k \geq K \quad G_k = G_K \quad \square$$