Zariski Theorem

Suppose $k \subset K$

K is f. generated over k

as algebra. Then K

is algebraic over k.

Reminders

$x \in K$ is alg. over k

if it is a root of

polynomial $p(t) \in k[t]$

Equivalently:

$(1, x, x^2, x^3, \ldots)$

span a finite-dimensional

k-vector space

Suppose $K = \langle f_1, f_2, \dots f_n \rangle$

as k-algebra.

$k \subset k(f_1) \subset k(f_1, f_2) \subset \dots \subset k(f_1 \dots f_n) = K$

Take the last f_i

which is <u>not</u> algebraic

over $k(f_1, \dots, f_{i-1})$

If K is not alg/k

then this f_i exists.

Suppose K is not

algebraic over k.

Replace k by

$k(f_1, \dots, f_{i-1})$

K is not algebraic over new K:

$k(f_i) \cong k(t)$

But: 1) K is generated over new k as algebra by f. many elements.

2) $[K : k(f_i)] < \infty$

Suppose $\{e_1, \dots e_N\}$ is a basis of K over $k(f_i)$

We can assume $e_1 = 1$

As a set K is isomorphic to $k(t)^N$

$\forall i, j \in \{1, \dots N\}$

$$e_i \cdot e_j = \sum_{k=1}^{N} b_{ijk} e_k$$

$$b_{ijk} \in k(t)$$

K is generated as k-algebra by f. many f_i

$$f_i = \sum \underset{\substack{\cap \\ k(t)}}{c_{ik}} e_k$$

We are given that f_i generate K as algebra

The algebra generated by f_i is contained in the set of

$$\sum_{k=1}^{N} a_k e_k$$

s.t. the irreducible factors of denominators of a_k are among the irreducible factors of denominators of b_{ij}'s and c_{jk}

But $\exists$ more irreducible factors in $K[t]$.

So $\frac{1}{p(t)} e_1 \in K$

but not to $k(\{f_i\})$

Contradiction. So K is algebraic $/k$.

$\square$

Thm. If $m \subset \mathbb{C}[x_1 \ldots x_n]$ is a maximal ideal, then

$m = (x - a_1, x - a_2, \ldots x - a_n)$

Proof. $\mathbb{C}[x_1 \ldots x_n]/m$ is f.g. over $\mathbb{C}$ <u>as algebra</u> by $\bar{x}_1, \ldots, \bar{x}_n$ $\quad \bar{x}_i = f(x_i)$

$$f : \mathbb{C}[x_1 \ldots x_n] \to \mathbb{C}[x_1 \ldots x_n]/m$$

By ~~Zariski~~ Zariski theorem,

$$\mathbb{C}[x_1 \ldots x_n]/m \cong \mathbb{C}$$

Define $a_i = f(x_i)$

under this identification.

$\forall p \in \mathbb{C}[x_1 \ldots x_n]$

$f(p) = p(a_1, \dots, a_n)$

$m = \{p \mid p(a_1 \dots a_n) = 0\}$

p in powers of $x_1 - a_1$,

$x_2 - a_2$,

$\dots$

$x_n - a_n$

$m = \langle x_1 - a_1, \dots, x_n - a_n \rangle$

Weak Nullstellensatz.

If $f_1, \dots, f_N \in \mathbb{C}[x_1 \dots x_n]$

have no common zeroes

then $1 \in \langle f_1, \dots, f_N \rangle$

(or $\langle f_1 \dots f_N \rangle = \mathbb{C}[x_1 \dots x_n]$)

Hilbert's Nullstellensatz

(Rabinowitsch trick)

Suppose $f_1 \ldots f_N \in \mathbb{C}[x_1 \ldots x_n]$

Suppose $f_{\cancel{N}+1}$ that

vanishes at all common

roots of $f_1 \ldots f_N$.

Then $\exists\, m \geq 1$

$$f_{N+1}^m \in \langle f_1, \ldots, f_N \rangle$$

Proof (Rabinowitsch trick)

In $\mathbb{C}[x_1, x_2, \ldots, x_n, x_{n+1}]$

consider

$\langle f_1, \dots f_N, 1 - f_{N+1} \cdot x_{n+1} \rangle$

If all of these are 0 at $(a_1, \dots, a_n, a_{n+1})$

then $f_1, \dots, f_N$ vanish

at $(a_1, \dots, a_n)$

So $f_{N+1}(a_1, \dots a_n) = 0$

So $1 - f_{N+1} \cdot x_{n+1} = 1$

By weak Nullstellensatz

$\exists\, p_1, \dots p_{N+1} \in \mathbb{C}[x_1 \dots x_{n+1}]$

$f_1 p_1 + \dots + f_N p_N + (1 - f_{N+1} \cdot x_{n+1}) p_{N+1} = 1$

$$\mathbb{C}[x_1 \dots x_{n+1}] \to \mathbb{C}(x_1, \dots, x_n)$$

s.t. $\quad x_1 \mapsto x_1, \dots, x_n \mapsto x_n$

$$x_{n+1} \mapsto \frac{1}{f_{N+1}(x_1 \dots x_n)}$$

$$x_{n+1} \mapsto \frac{1}{f_{N+1}}$$

$$f_1 \cdot \frac{\tilde{p}_1(x_1 \dots x_n)}{\underline{f_{N+1}^{\alpha_1}}} + \dots + f_N \cdot \frac{\tilde{p}_N}{\underline{f_N^{\alpha_N}}} = 1$$

$$f_1 \cdot \tilde{p}_1(x_1 \dots x_n) \cdot f_{N+1}^{?} + \dots + f_N \tilde{p}_N \cdot f_{N+1}^{?} = f_{N+1}^{m}$$

$P_1 = x_1^2 + x_{n+1}$

$\Downarrow$

$$x_1^2 + \frac{1}{f_{N+1}} = \frac{x_1^2 f_{N+1} + 1}{f_{N+1}}$$

<u>Def.</u> An affine algebraic variety over $\mathbb{C}$ is $m\mathrm{Spec}\left(\mathbb{C}[x_1, \ldots x_n]/I\right)$ where $I = \mathrm{rad}(I)$.

<u>Def.</u> An algebraic affine set is a set of solutions of a system of equations in $x_1 \ldots x_n$

Algebraic maps.

$$X_1 \xrightarrow{f} X_2$$

R_1

$$X_1 = \mathrm{mSpec}\,(\underbrace{\mathbb{C}[x_1 \dots x_n]/I}_{})$$

$$X_2 = \mathrm{mSpec}\,(\underbrace{\mathbb{C}[y_1 \dots y_m]/J}_{R_2})$$

A map $f : X_1 \to X_2$

is a ring homomorphism

$$R_2 \xrightarrow{f^*} R_1$$

$$\begin{tikzcd} \mathbb{C}[y_1 \dots y_m] \arrow[r, dashed] \arrow[d] \arrow[dr] & \mathbb{C}[x_1 \dots x_n] \arrow[d] \\ R_2 \arrow[r, "f^*"] & R_1 \end{tikzcd}$$