More on varieties

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, phantom, "\cap" rotate=90] & X_2 \arrow[d, phantom, "\cap" rotate=90] \\ \mathbb{C}^n \arrow[r, dashed] & \mathbb{C}^m \end{tikzcd}$$

$$X_1 = \mathrm{mSpec}\left(\mathbb{C}[x_1 \ldots x_n]/I\right)$$

$$X_2 = \mathrm{mSpec}\left(\mathbb{C}[y_1 \ldots y_m]/J\right)$$

$$\begin{tikzcd} \mathbb{C}[x_1 \ldots x_n]/I & \mathbb{C}[y_1 \ldots y_m]/J \arrow[l, "f^*"'] \\ \mathbb{C}[x_1 \ldots x_n] \arrow[u] & \mathbb{C}[y_1 \ldots y_m] \arrow[l, dashed, "\varphi^*"'] \arrow[u] \arrow[ul] \end{tikzcd}$$

Ideal $\mathfrak{m} \subset \mathbb{C}[x_1 \ldots x_n]/I$

maximal

$$\mathfrak{m} \subset \mathbb{C}[x_1 \ldots x_n]$$

$$\parallel$$

$$(x_1 - a_1, \ldots x_n - a_n)$$

$$\mathfrak{m} \rightsquigarrow (f^*)^{-1}(\mathfrak{m})$$

$\varphi^*(y_i)$ is a polynomial in $x_1 \ldots x_n$ $\quad i = 1, \ldots m$

$$(\varphi^*)^{-1}(x_1 - a_1, \ldots, x_n - a_n)$$

$$\parallel$$

$$(y_1 - \varphi^*(y_1)(a_1 \ldots a_n), \ldots, y_m - \varphi^*(y_m)(a_1 \ldots a_n))$$

<u>Ex.</u> $\mathbb{C}[x] \xhookrightarrow{\alpha} \mathbb{C}(x)$

(0) maximal

$\alpha^{-1}(0) = (0)$ prime, not maximal

Ex. $X_1 = \mathbb{C} = \mathrm{mspec}(\mathbb{C}[t])$

$X_2 = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 + 1\}$

$\mathrm{mSpec}(\mathbb{C}[x,y]/(y^2 - x^3 - 1))$

There is no rational map

$X_1 \to X_2$ (non-constant)

Thm. $\nexists$ $\dfrac{p(t)}{q(t)}, \dfrac{r(t)}{s(t)}$

(rational, non-constant)

s.t. $\left(\dfrac{p(t)}{q(t)}\right)^2 = \left(\dfrac{r(t)}{s(t)}\right)^3 + 1$

Proof Suppose

$$\frac{p^2}{q^2} = \frac{r^3}{s^3} + 1$$

$$\frac{p^2 - q^2}{q^2} = \frac{r^3}{s^3}$$

$$q^2 = s^3 \cdot \text{const}$$

$$q = u^3, \quad s = u^2$$

$$\frac{p^2 - u^6}{u^6} = \frac{r^3}{u^6}$$

$\gcd(p, u) = 1$

$\gcd(p, r) = 1$

$\gcd(u, r) = 1$

$$\begin{cases} p^2 - u^6 = r^3 \\ 2pp' - 6u^5u' = 3r^2r' \end{cases}$$

$$D = \begin{vmatrix} p^2 & u^6 \\ 2pp' & 6u^5u' \end{vmatrix} = \begin{vmatrix} r^3 & u^6 \\ 3r^2r' & 6u^5u' \end{vmatrix}$$

$p \mid D, \quad u^5 \mid D, \quad r^2 \mid D$

$pu^5r^2 \mid D$

Suppose first: $\deg p^2 = \deg u^6 = \deg r^3$
$$\underset{m}{\parallel}$$

$$\deg D \le m + (m-1) = 2m - 1$$

$\deg p = \frac{1}{2} m$

$\deg u = \frac{1}{6} m$

$\deg r = \frac{1}{3} m$

$\deg(pu^5r^2) = \frac{1}{2} m + \frac{5}{6} m + \frac{2}{3} m$

$= 2m$

So $D = 0$:

$p^2 \cdot 6u^5 u' - u^6 \cdot 2pp' = 0$

$6pu' = 2up'$

$\begin{cases} \gcd(p, u) = 1 \\ p \mid up' \end{cases} \Rightarrow p \mid p'$

(Contradiction)

Suppose that one of p^2, u^6, r^3 has degree $m - n$. $n \geq 1$

Then

$$\deg(D) \le m + m - n - 1$$
$$= \underline{2m - n - 1}$$

$$\deg(pu^5r^2) = \begin{cases} \frac{1}{2}(m-n) + \frac{5}{6}m + \frac{2}{3}m \\ \frac{1}{2}m + \frac{5}{6}(m-n) + \frac{2}{3}m \\ \frac{1}{2}m + \frac{5}{6}m + \frac{2}{3}(m-n) \end{cases}$$

$$\deg(pu^5r^2) = 2m - \begin{cases} \frac{1}{2}n \\ 5/6\, n \\ 2/3\, n \end{cases}$$

$$\ge \underline{2m - n}$$

$\square$

abc-Theorem

Stothers-Mason

Two examples

1) $\mathbb{CP}^1$ $\quad \mathbb{C}, \mathbb{C}$ glued

by $\mathbb{C}^*$

$\mathbb{C}[t], \quad \mathbb{C}[x]$

$x = \frac{1}{t} \quad t = \frac{1}{x}$

$$\mathbb{C}[t] \hookrightarrow \mathbb{C}[t, \tfrac{1}{t}]$$

$$\cong \quad x = \frac{1}{t}$$

$$\mathbb{C}[x] \hookrightarrow \mathbb{C}[x, \tfrac{1}{x}]$$

2) $\mathbb{C}^2 \setminus \{(0,0)\}$

not an affine variety

(Hartog's Extension Thm)

$\mathbb{C}^2 \setminus \{x=0\} = \text{mSpec}(\mathbb{C}[x, \frac{1}{x}, y])$

$\mathbb{C}^2 \setminus \{y=0\}$

Affine toric variety

Suppose $\mathbb{Z}^n \subseteq \mathbb{R}^n$,

σ convex cone in $\mathbb{R}^n$

σ rational, polyhedral, closed

1) $\forall \vec{x} \in \sigma, \lambda \in \mathbb{R}_{\geq 0}$

$\Downarrow$

$\lambda \vec{x} \in \sigma$

2) convex as a set

$\Leftarrow$) $\forall \vec{x}_1, \vec{x}_2 \in \sigma$ |

$\vec{x}_1 + \vec{x}_2 \in \sigma$

polyhedral:

$\sigma = \mathrm{span}(\vec{x}_1, \ldots, \vec{x}_N)$

rational:

$\vec{x}_1, \ldots, \vec{x}_N \in \mathbb{Q}^n$

$\mathbb{Z}^n$

$X_\sigma = \mathrm{mspec}(\underline{\mathbb{C}[\sigma \cap \mathbb{Z}^n]})$

$\sigma \cap \mathbb{Z}^n$ f.g.

semigroup

$G \qquad \mathbb{C}[G]$