Affine Toric Varieties

Cone $\sigma \subset \mathbb{R}^n$

σ convex rational polyhedral closed cone

$$\mathrm{mSpec}\left(\mathbb{C}[\sigma \cap \mathbb{Z}^n]\right)$$

Gordan's Lemma

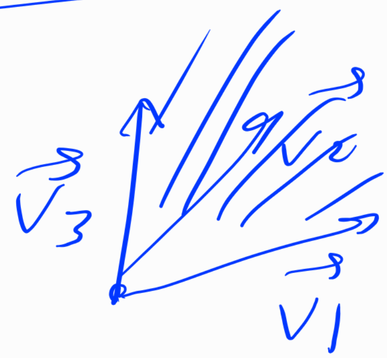
Suppose σ is closed convex rational polyhedral cone. Then $\sigma \cap \mathbb{Z}^n$ is finitely generated as a semigroup

Proof. Case:

1) $\mathrm{Span}(\sigma) = \mathbb{R}^n$

2) $\sigma = \langle \vec{v}_1, \ldots, \vec{v}_n \rangle$

$\vec{v}_i \in \mathbb{Q}^n$



First, we may assume that $\forall i \ \vec{v}_i \in \mathbb{Z}^n$

$$\sigma = \left\{ \sum_{i=1}^{n} x_i \vec{v}_i \;\middle|\; x_i \geq 0 \right\}$$

$$\vec{v} = \sum_{i=1}^{n} x_i \vec{v}_i = \sum_{i=1}^{n} \lfloor x_i \rfloor \vec{v}_i \in \mathbb{Z}^n + \sum_{i=1}^{n} \{x_i\} \vec{v}_i$$

$$S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \cup X$$

$$X = \mathbb{Z}^n \cap \left\{ \sum x_i \vec{v}_i \;\middle|\; \forall x_i \in [0,1] \right\}$$

General Case

$\sigma = \langle \vec{v}_1, \ldots, \vec{v}_N \rangle$

The same argument as above works whenever $\vec{v}_1 \ldots \vec{v}_N$ are linearly independent.

Lemma. $\forall\, \vec{v} = \sum_{i=1}^{N} \lambda_i \vec{v}_i,\ \lambda_i \geq 0$

$\exists$ a linearly independent collection $\vec{v}_{i_1}, \ldots \vec{v}_{i_k}$ s.t. $\vec{v}$ is a non-negative linear combination of $\vec{v}_{i_1} \ldots \vec{v}_{i_k}$

Proof of Lemma.

Suppose $\vec{v} = \sum_{i=1}^{N} \lambda_i \vec{v_i}$, $\forall i \; \lambda_i > 0$

and

$\sum c_i \vec{v_i} = 0$. (not all $c_i = 0$)

Then we assume that at least one of $c_i > 0$.

$$\vec{v} = \sum_{i=1}^{N} \lambda_i \vec{v_i} - x \sum_{i=1}^{N} c_i \vec{v_i}$$

$$= \sum (\lambda_i - x c_i) \vec{v_i}$$

Take $x_0 = \max\{x \mid \forall i \; \lambda_i - x c_i \geq 0\}$

At this x_0 for one (or more) i $\lambda_i - x_0 c_i = 0$.

So $\vec{v} = \sum_{\substack{j \neq i \\ j=1}}^{N} (\lambda_j - x_0 c_j) \vec{v_j}$

$$\vec{v} = \vec{v}_1 + 2\vec{v}_2 + \vec{v}_3$$

$$\vec{v}_1 + 10\vec{v}_2 - \vec{v}_3 = \vec{0}$$

$$\vec{v} = \vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 - x(\vec{v}_1 + 10\vec{v}_2 - \vec{v}_3)$$

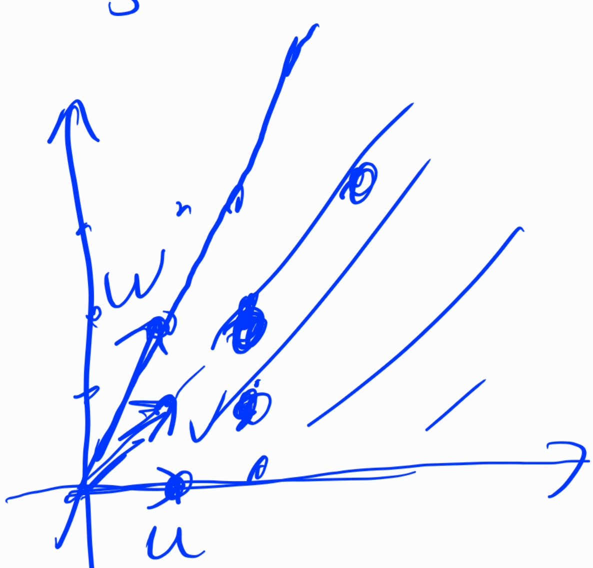
$$= (1-x)\vec{v}_1 + (2-10x)\vec{v}_2 + \underline{(1+x)\vec{v}_3}$$

$$(1-x) \geq 0 \qquad (2-10x) \geq 0$$

$$x \leq 1 \qquad x \leq \frac{1}{5} \qquad x_0 = \frac{1}{5}$$

$$\vec{v} = \left(1-\frac{1}{5}\right)\vec{v}_1 + 0\cdot\vec{v}_2 + \left(1+\frac{1}{5}\right)\vec{v}_3$$

$$\vec{v} = \frac{4}{5}\vec{v}_1 + \frac{6}{5}\vec{v}_3 \quad \square$$

Ex. 

$$\left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\rangle$$

$\mathbb{Z}^2 \cap \sigma$ is generated as a semigroup by $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$

$$\mathbb{C}[\mathbb{Z}^2 \cap \sigma] \cong \underline{\mathbb{C}[u,v,w]/(uw-v^2)}$$

In the lattice

$$\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix}$$

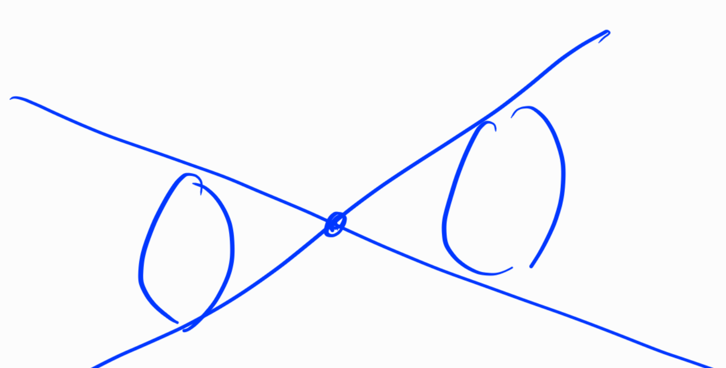
$$u \cdot w = v^2$$

$$\begin{bmatrix}4\\4\end{bmatrix} = 4\begin{bmatrix}1\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}1\\2\end{bmatrix}$$

$$= 2\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$$

$$v^4 = v^2 \cdot uw = u^2w^2$$

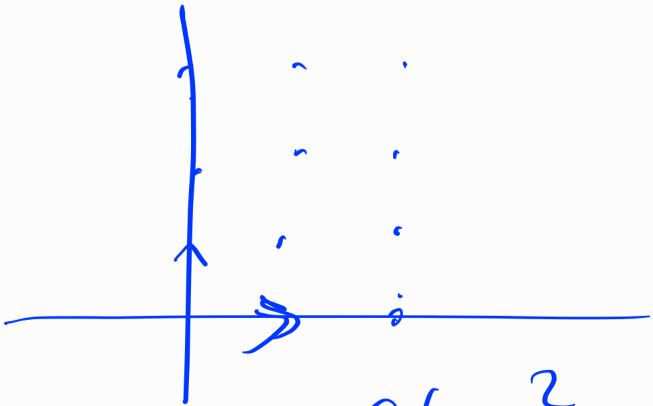
A cone



$$v^2 = uw$$

$$4v^2 = (u+w)^2 - (u-w)^2$$

Ex.



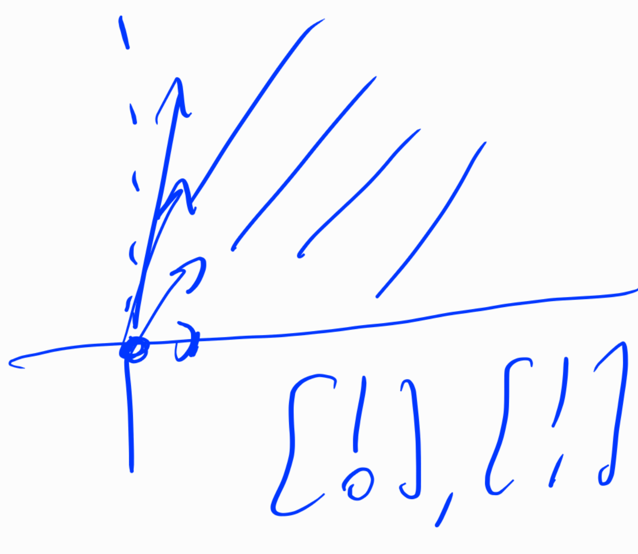
$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$$\mathbb{C}(\mathbb{Z}^2 \cap \sigma) = \mathbb{C}[x,y]$$

$$m\mathrm{Spec}(\mathbb{C}[x,y]) \cong \mathbb{C}^2$$

Why closed, rational, polyhedral

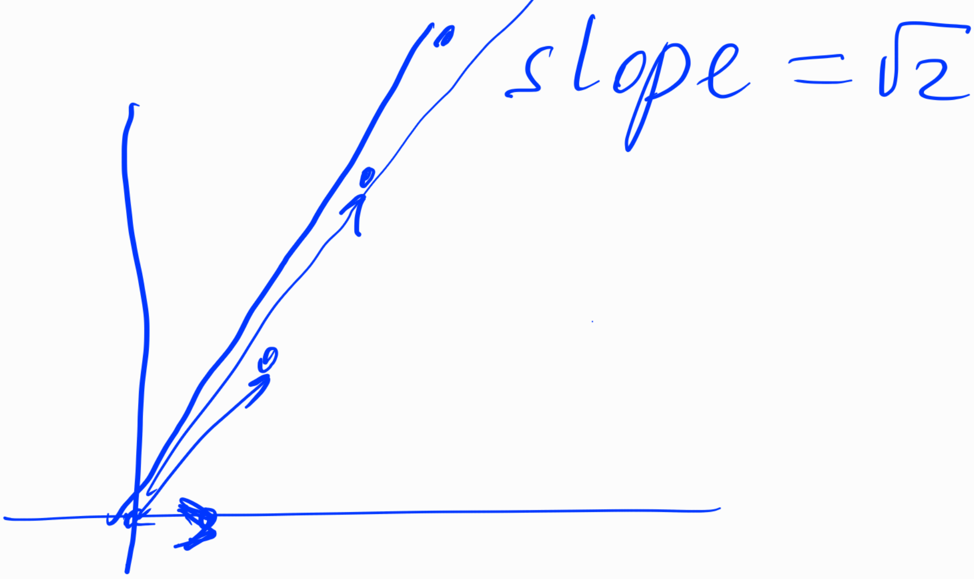
Ex.



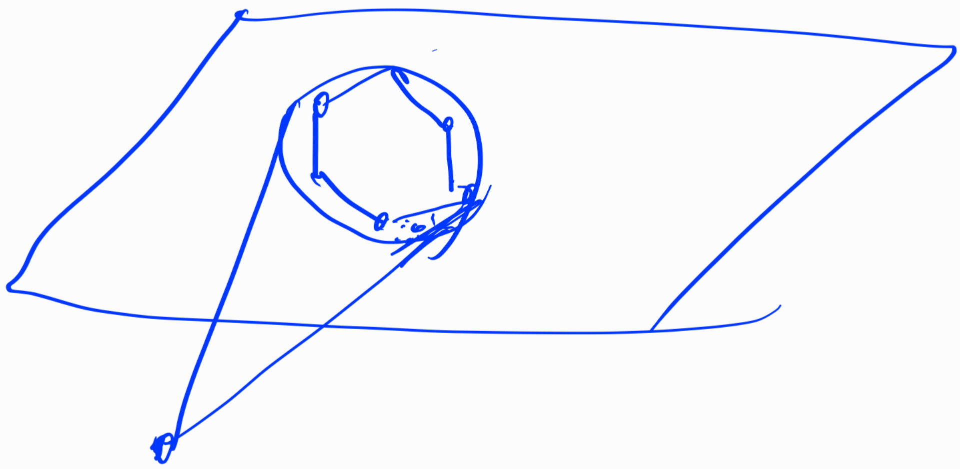
$\sigma = \left\{\begin{bmatrix}x\\y\end{bmatrix} \middle| x>0, y\geq 0\right\}$ and $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \dots, \begin{bmatrix}1\\n\end{bmatrix}, \dots$

Ex.



Ex.



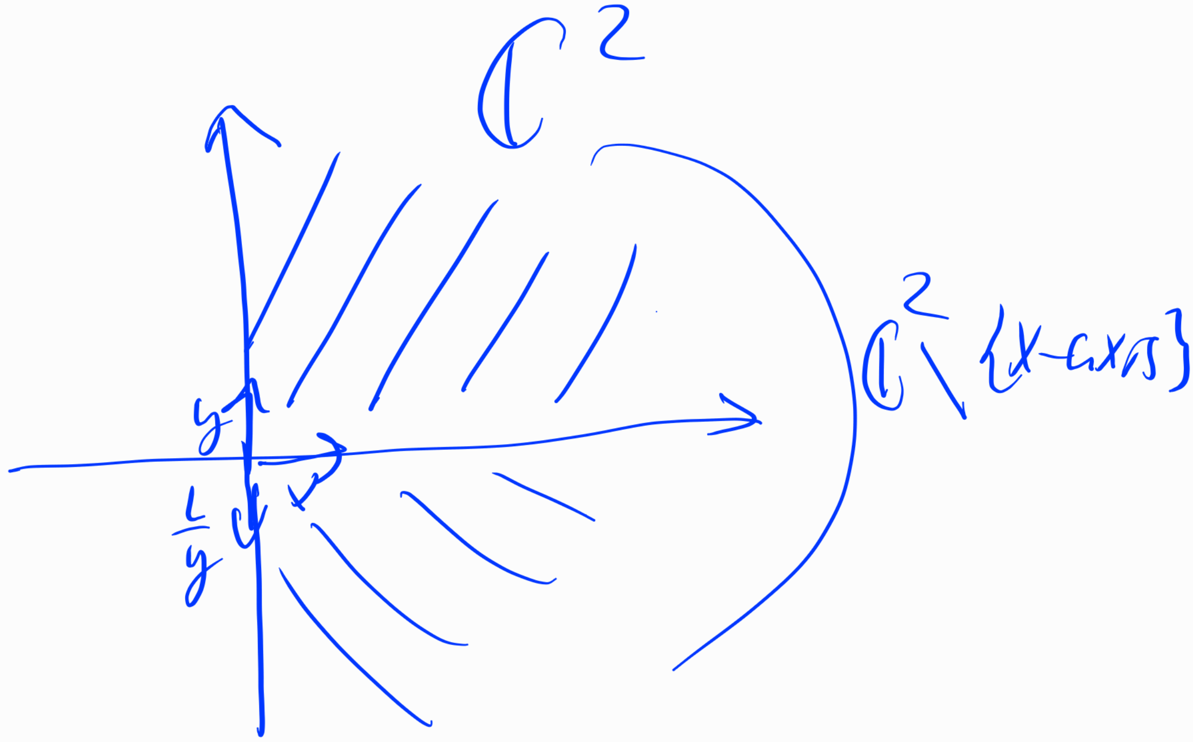
Thm. If the span(σ) is a rational subspace of $\mathbb{R}^n$ then iff. $\sigma \cap \mathbb{Z}^n$ is f.g. semigroup then σ is closed, rational, polyhedral

Ex. $\mathbb{C}[x,y]$ $\qquad$ $\mathbb{C}^2$

$y \neq 0$

$\mathbb{C}[x,y,\frac{1}{y}]$ $\qquad$ $\mathbb{C}^2 \setminus \{x\text{-axis}\}$

localization of $\mathbb{C}[x,y]$ by $\{1, y, y^2, \dots\}$



$$\mathbb{C}^2 \setminus \{x\text{-axis}\} = \mathbb{C} \times \mathbb{C}^*$$

$$\mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y, \tfrac{1}{y}]$$

Duality of convex cones

$V \qquad V^* = \mathrm{Hom}(V, \mathbb{R})$

If we have a non-degenerate quadratic form on V,

$$V^* \cong V$$
$$\begin{array}{c} \Psi \\ f \rightsquigarrow \vec{v}_f \end{array}$$
$$f(\vec{v}) = (\vec{v}, \vec{v}_f)$$

$$\sigma \subseteq V$$

$$\hat{\sigma} \subseteq V^*$$

$$\hat{\sigma} = \{ f \in V^* \mid \forall \vec{v} \in \sigma \;\; f(\vec{v}) \geq 0 \}$$

$$\hat{\sigma} = \{\vec{w} \in \mathbb{R}^n \mid \forall \vec{v} \in \sigma \ \ \vec{v} \cdot \vec{w} \geq 0\}$$