Duality for cones, vector spaces, lattices

Vector spaces

$V / \mathbb{R}$

$V^* = \mathrm{Hom}(V, \mathbb{R})$

If V is f.d. with basis $\{\vec{v}_1, \dots, \vec{v}_n\}$

then $\exists$ "dual basis" of V^*: $\vec{v}_1^{\,*}, \dots, \vec{v}_n^{\,*}$

$$\vec{v}_i^{\,*}(\vec{v}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\vec{v}_i^{\,*}(a_1\vec{v}_1 + \dots + a_n\vec{v}_n) = a_i$$

$(V^*)^* \cong V \quad (\dim V < \infty)$

If $\vec{v} \in V$, then $\forall f \in V^*$ we can consider $f(\vec{v})$

For a fixed $\vec{v}$, it is linear in f, which allows us to consider $\vec{v}$ as an element of $(V^*)^*$.

$$\alpha : V \longrightarrow (V^*)^*$$

$$\alpha(\vec{v})(f) \overset{\text{def}}{=} f(v)$$

α is linear, bijection

Dual basis to dual basis is the original basis.

If $V = \mathbb{R}^n$ we can identify V^* with $\mathbb{R}^n$ as well.

To identify V, V^* it is enough to fix a non-degenerate quadratic form. (e.g. a positive definite quadratic form).

Every $f \in Hom(V, \mathbb{R})$ is of the form

$$f(\vec{v}) = (\vec{v}, \vec{a})$$

$\vec{a}$ depends on f.

That $\vec{a}$ is unique

$\mathbb{Z}^n \subseteq \mathbb{R}^n$

Cones. only closed, convex

Def. σ is a closed convex cone in $\mathbb{R}^n$ if

1) σ is a closed subset, not empty

2) $\forall \vec{v} \in \sigma$, $\lambda \in \mathbb{R}_{\geq 0}$ $\lambda \vec{v} \in \sigma$

3) $\forall \vec{v_1}, \vec{v_2} \in \sigma$ $\underline{\vec{v_1} + \vec{v_2} \in \sigma}$

($\Leftarrow$) $\forall \vec{v_1}, \vec{v_2} \in \sigma$, $\forall \lambda \in [0,1]$ $\lambda \vec{v_1} + (1-\lambda)\vec{v_2} \in \sigma$)

Def. The span of σ is its span as a subset of $\mathbb{R}^n$.

Def. The co-span of σ is $\sigma \cap (-\sigma)$

i.e. $\{\vec{v} \in \sigma \mid -\vec{v} \in \sigma\}$

Thm. $\mathrm{cospan}(\sigma)$ is a subspace of $\mathbb{R}^n$.

Proof ... $\square$

Note. Every subspace of $\mathbb{R}^n$ is a cone in $\mathbb{R}^n$

$$\sigma = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, x_2 \geq 0,\ x_3 \geq 0 \right\}$$

$$\mathrm{cospan}(\sigma) = \mathbb{R} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Def. The dual of $\sigma \subset V (= \mathbb{R}^n)$ is the set of all $f \in V^* (= \mathbb{R}^n)$ s.t. $f(\vec{v}) \geq 0 \ \forall \vec{v} \in \sigma$

Ex.

$\subset \mathbb{R}^2$

$$\boxed{\hat{\hat{\sigma}} = \sigma}$$

Ex. $W \subset \mathbb{R}^n$ is a subspace. Then

$$\hat{W} = W^{\perp}$$

Proof. $f \in \hat{W}$ iff

$\forall \vec{v} \in W \quad f(\vec{v}) \geq 0$

$$\begin{cases} f(\vec{v}) \geq 0 \\ f(-\vec{v}) \geq 0 \end{cases} \Longleftrightarrow f(\vec{v}) = 0$$

$$f(-\vec{v}) = -f(\vec{v})$$

For every cone σ

$$\mathrm{Cospan}(\sigma) \subseteq \sigma \subseteq \mathrm{Span}(\sigma)$$

$$\mathrm{Span}(\sigma)^{\perp} \subseteq \hat{\sigma} \subseteq \mathrm{Cospan}(\sigma)^{\perp}$$

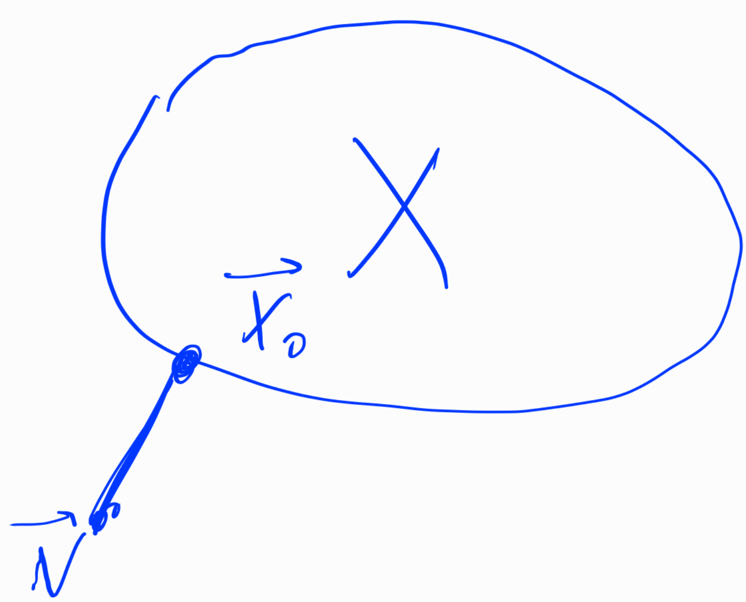
$\mathrm{Span}(\sigma)^{\perp} = \mathrm{Cospan}(\hat{\sigma})$ ← To be proved → $\mathrm{Cospan}(\sigma)^{\perp} = \mathrm{span}(\hat{\sigma})$

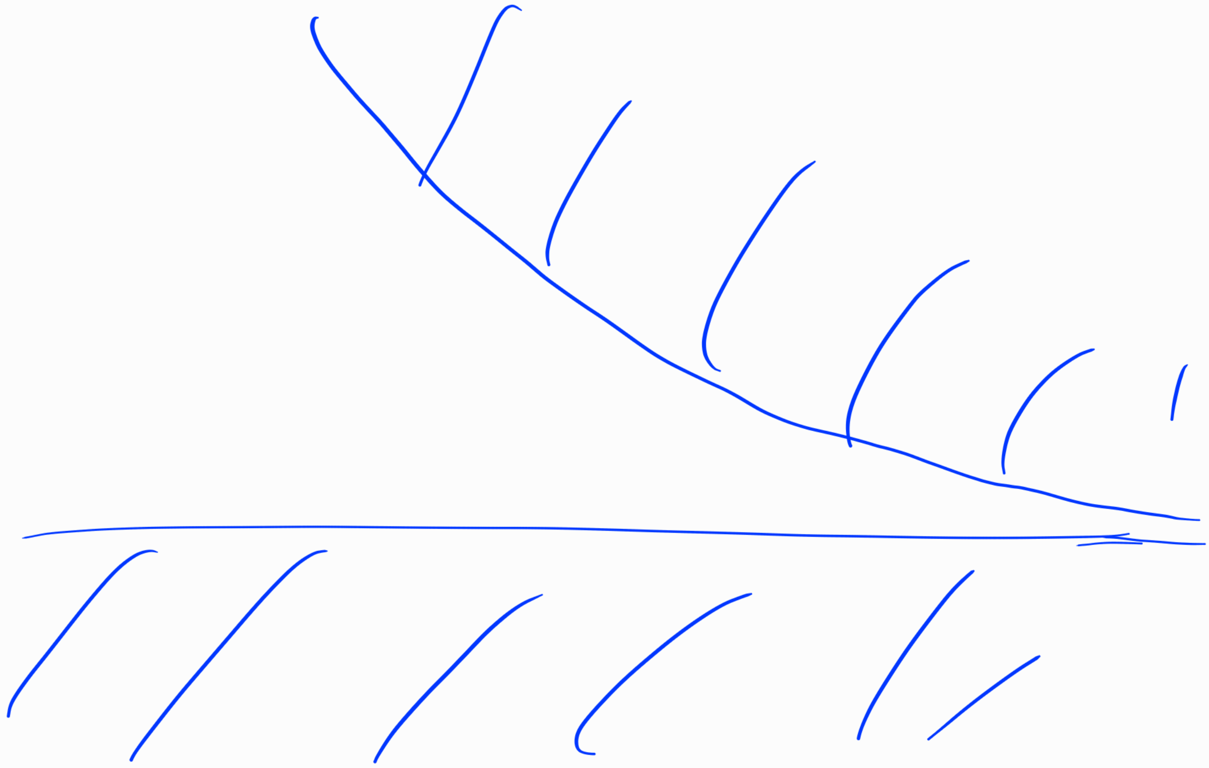
Harder Thm

$$\hat{\hat{\sigma}} = \sigma$$

Lemma. Suppose $X \subset \mathbb{R}^n$ is any closed convex set. Suppose $\vec{v} \in \mathbb{R}^n$. Then $\exists!\ \vec{x}_0 \in X$ s.t.

$$d(\vec{x}_0, \vec{v}) = \min_{\vec{x} \in X} d(\vec{x}, \vec{v})$$





Proof #1.

Suppose $\inf_{\vec{x} \in X} (d(\vec{x}, \vec{v})) = d.$

Consider $\vec{x}_1, \vec{x}_2, \ldots$

s.t. $d(\vec{x}_i, \vec{v}) \le d + \frac{1}{i}$

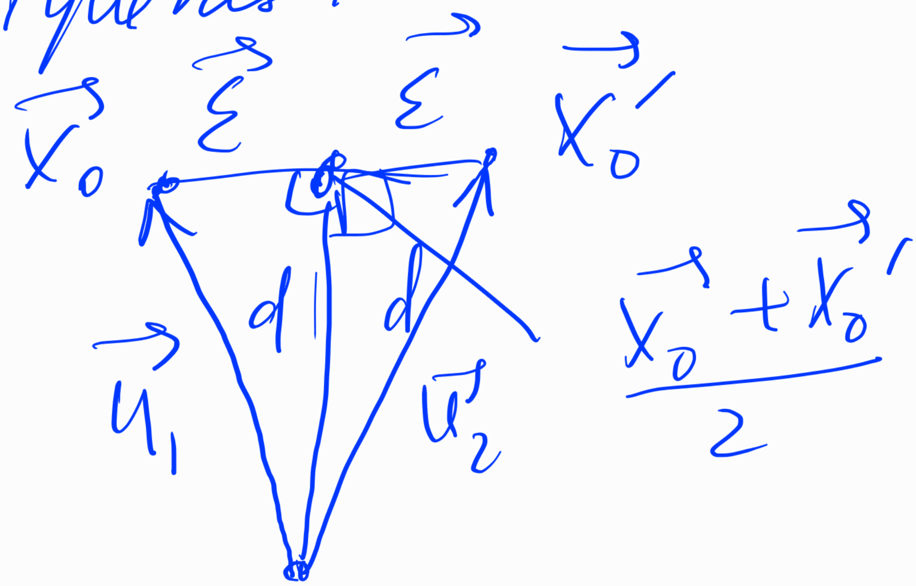
$\{\vec{x}_i\}$ is bounded

So $\exists$ a subsequence s.t.

it converges to $\vec{x}_0$.

$d(\vec{x}_0, \vec{v}) = d$, by continuity of d.

Uniqueness:



$\vec{u}$ $\vec{v}$

$\|$

$$\left(\frac{\vec{u}_1+\vec{u}_2}{2}, \frac{\vec{u}_1+\vec{u}_2}{2}\right)$$

$$(\underbrace{\vec{u}_2-\vec{u}_1}_{2\vec{\varepsilon}}, \underbrace{\vec{u}_2-\vec{u}_1}_{2\vec{\varepsilon}}) > 0$$

$$(\vec{u}+\vec{\varepsilon}, \vec{u}+\vec{\varepsilon}) = d^2$$

$$(\vec{u}-\vec{\varepsilon},\ \vec{u}-\vec{\varepsilon}) = d^2$$

$$2(\vec{u}\,\vec{u}) + \underline{2(\vec{u}\cdot\vec{\varepsilon}) - 2(\vec{u}\cdot\vec{\varepsilon})} + 2\vec{\varepsilon}\cdot\vec{\varepsilon} = 2d^2$$

$$(\vec{u},\vec{u}) = (\vec{d},\vec{d}) - (\vec{\varepsilon},\vec{\varepsilon}) < (\vec{d},\vec{d})$$

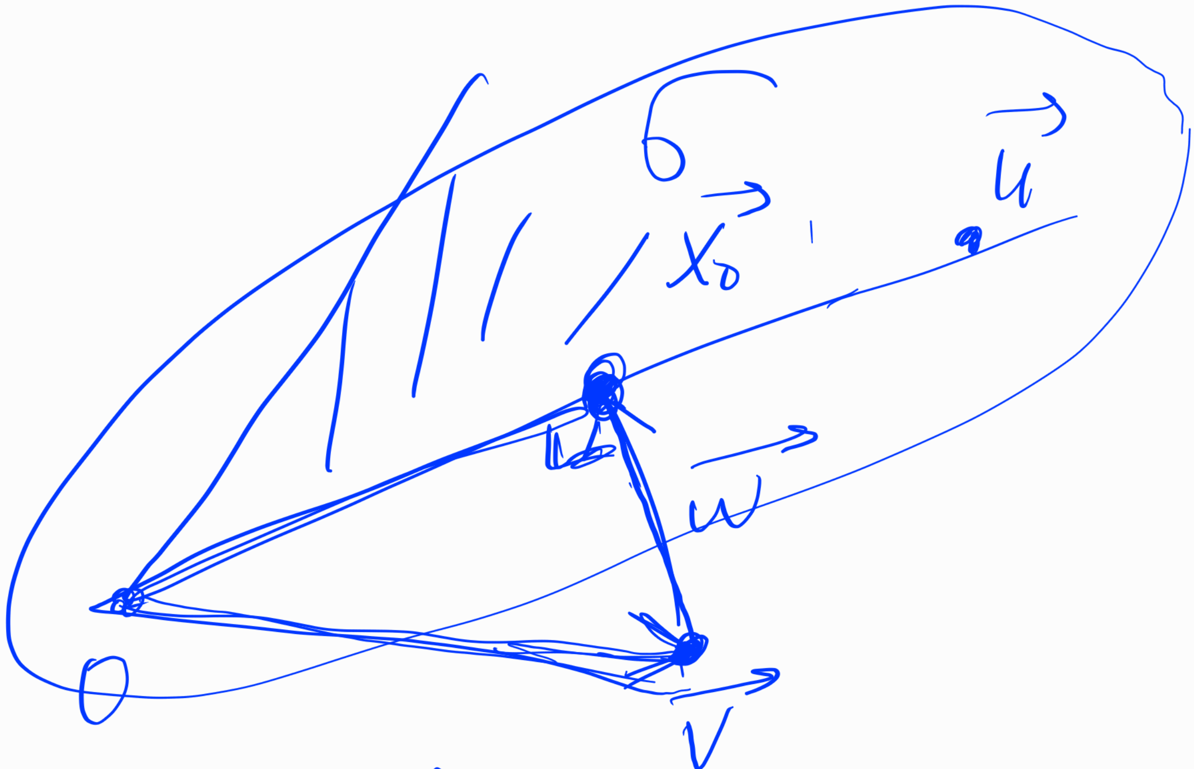
<u>Thm.</u> $\hat{\hat{\sigma}} = \sigma$

<u>Proof.</u> $\sigma \subseteq \hat{\hat{\sigma}}$ <u>obvious</u>

We need: if $\vec{v} \notin \sigma$

then $\exists\, \vec{w} \in \hat{\sigma}$ s.t.

$$(\vec{v},\vec{w}) < 0$$



Claim. $\vec{w} \in \hat{\sigma}$:

If $u \in \sigma$, s.t. $(\vec{u}, \vec{w}) < 0$

then $\vec{x}_0 + t\vec{u}$, t small positive

distance $d(\vec{x}_0 + t\vec{u}, \vec{v}) < d$.

So $\vec{w} \in \hat{\sigma}$

But $(\vec{w} \cdot \vec{v}) < 0$

Proof #2 of Lemma.

X convex closed $\quad \vec{v} \in V$

Then $\exists!\ x_0 \in X$ s.t.

$$d(\vec{x}_0, \vec{v}) = \min_{\vec{x} \in X} (d(\vec{x}, \vec{v}))$$

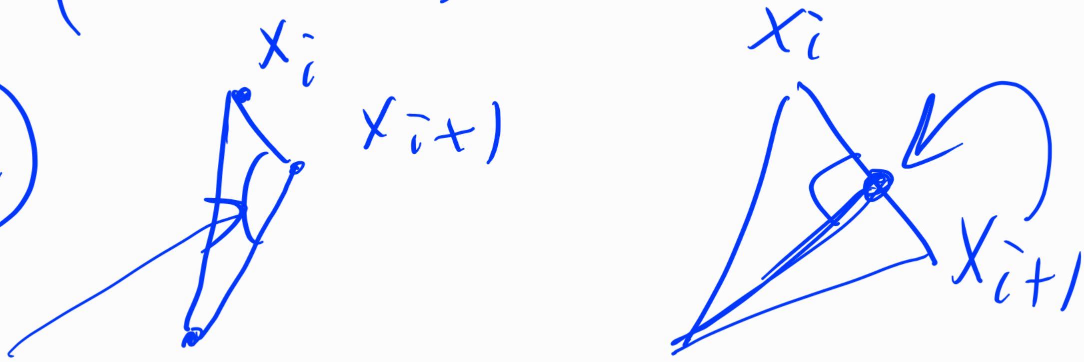
Proof uses convexity instead of compactness.

$$d = \inf_{\vec{x} \in X} \{ d(\vec{x}, \vec{v})\}$$

$\vec{x}_i \in X$

1) $\left(d(\vec{x}_i, \vec{v})\right)^2 \le d^2 + \frac{1}{2^i}$

2)



angle V
is not acute