Thm. Suppose X is closed convex set in V, where V is Cauchy-complete Euclidean space.

Suppose $\vec{v} \in V$. Then

$\exists! \, \vec{x}_0 \in X$ s.t. $d(\vec{x}_0, \vec{v}) = \min_{\vec{x} \in X} d(\vec{x}, \vec{v})$

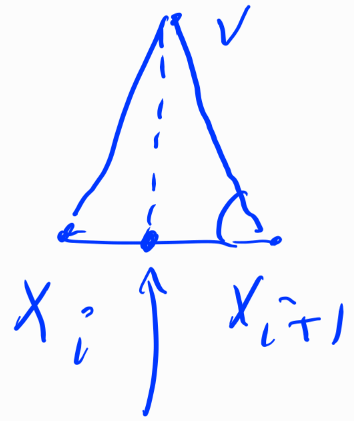
Proof. Suppose

$$\inf_{\vec{x} \in X} d(\vec{x}, \vec{v}) = d$$

Construct a sequence $\vec{x}_1, \vec{x}_2, \ldots$ in X

s.t. 1) $d(\vec{x}_i, v)^2 \leq d^2 + \frac{1}{2^i}$

2) The angle at $\vec{x}_{i+1}$ of the triangle $\vec{x}_i \vec{x}_{i+1} \vec{v}$ is <u>not</u> <u>acute</u> $\left(\geq \frac{\pi}{2}\right)$



$[x_i, x_{i+1}] \subseteq X$

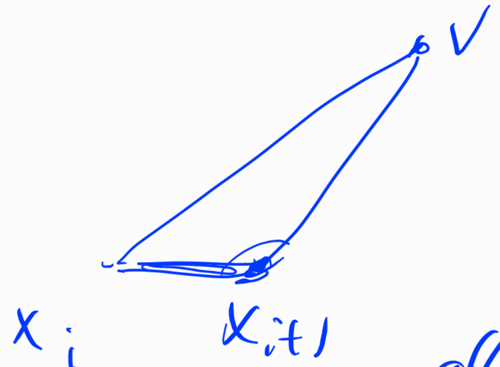
replace x_{i+1} by this.

<u>Claim</u> $\{x_i\}$ is a Cauchy sequence.

$$d(x_i, x_{i+1})^2 \le \underbrace{d(x_i, v)^2} - d(x_{i+1}, v)^2$$

$$\le \left(d^2 + \frac{1}{2^i}\right) - d^2$$

$$\le \frac{1}{2^i}$$



$\forall \varepsilon > 0$ for all large enough k, n $(k < n)$

$$|x_k - x_n| \le \sum_{i=k}^{n-1} |x_i - x_{i+1}|$$

$$\le \sum_{i=k}^{n} \frac{1}{(\sqrt{2})^i} < \frac{1}{(\sqrt{2})^k} \cdot \frac{1}{1 - 1/\sqrt{2}} < \varepsilon$$

$\lim \vec{x}_i = \vec{x}_0$

$d(\vec{x}_0, \vec{v}) = \lim d(\vec{x}_i, \vec{v}) = d.$ $\square$

__Corollary.__ If σ is a closed cone in V, then $\hat{\hat{\sigma}} = \sigma$.

__Corollary:__ If $W \subseteq V$ is a closed subspace, then $(W^{\perp})^{\perp} = W$. (V is Cauchy-complete)

$V = \mathbb{R}^n$, $\sigma \subset V$ closed polyhedral

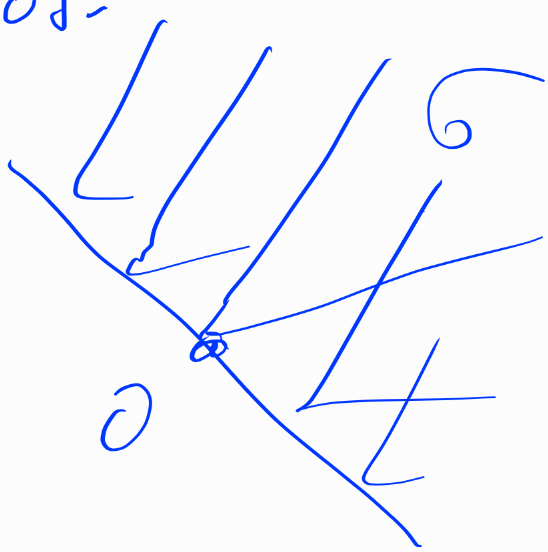
__Def.__ span(σ) is the smallest subspace of V that contains σ.

<u>Def.</u> Cospan(σ) is the largest subspace of V that is contained in σ.

(It is $\sigma \cap (-\sigma)$)

<u>Def.</u> σ is strictly convex if if cospan is $\{0\}$.

<u>Ex.</u>



<u>Not</u> strictly convex.

Operations on cones

1) $\sigma_1 + \sigma_2$

$= \{ \vec{v} \in V \mid \exists \vec{x}_1 \in \sigma_1, \exists \vec{x}_2 \in \sigma_2, \ \vec{v} = \vec{x}_1 + \vec{x}_2 \}$

2) $\underline{\sigma_1 \cap \sigma_2}$

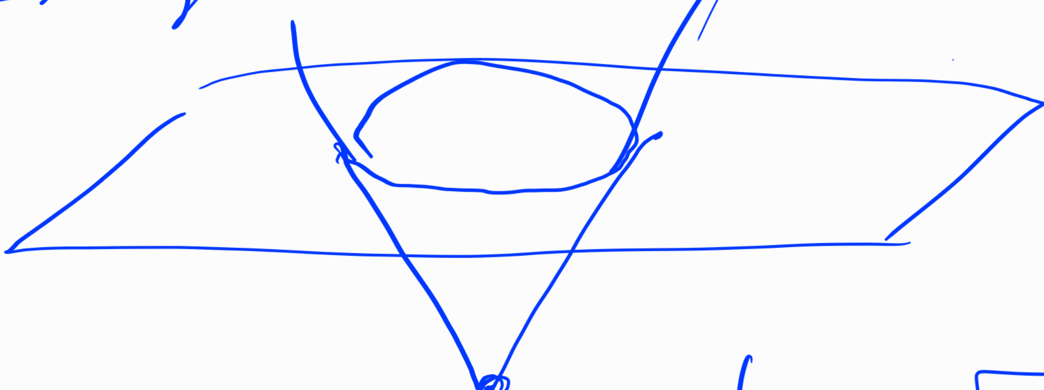
__Lemma.__ 1) $\widehat{\sigma_1 + \sigma_2} = \hat{\sigma}_1 \cap \hat{\sigma}_2$

2) $\widehat{\sigma_1 \cap \sigma_2} = \hat{\sigma}_1 + \hat{\sigma}_2$

$\supseteq$

__Proof.__

Example of non-polyhedral cone



$\mathbb{R}^3 \quad \sigma = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, z \geq \sqrt{x^2 + y^2} \right\}$

① $f \in V^* \quad \forall \vec{v}_1 \in \sigma_1, \vec{v}_2 \in \sigma_2$

$f(\vec{v}_1 + \vec{v}_2) \geq 0. \quad (f \in \widehat{\sigma_1 + \sigma_2})$

This implies by taking $\vec{v}_2 = 0$ that $f(\vec{v}_1) \geq 0 \ \forall \vec{v}_1 \in \sigma_1$

So $f \in \hat{\sigma}_1$.

Likewise $f \in \hat{\sigma}_2$.

So $f \in \hat{\sigma}_1 \cap \hat{\sigma}_2$.

But, in the other direction;

if $f \in \hat{\sigma}_1 \cap \hat{\sigma}_2$, then

$\forall \vec{v}_1 \in \sigma_1, \ \forall \vec{v}_2 \in \sigma_2$

$f(\vec{v}_1) \geq 0, \ f(\vec{v}_2) \geq 0$

So $f(\vec{v}_1 + \vec{v}_2) \geq 0$

2 $\widehat{\sigma_1 \cap \sigma_2} = \widehat{\hat{\tau}_1 \cap \hat{\tau}_2} \overset{1}{=}$

$\sigma_1 = \hat{\tau}_1, \ \sigma_2 = \hat{\tau}_2$

$= \widehat{\widehat{\tau_1 + \tau_2}} = \tau_1 + \tau_2 = \hat{\sigma}_1 + \hat{\sigma}_2$ $\square$

<u>Lemma</u>. If σ is polyhedral, then $\hat{\sigma}$ is also polyhedral.

<u>Proof</u>. $\sigma = \langle \vec{v}_1, \dots, \vec{v}_N \rangle$ as a cone

$$\hat{\sigma} = \{ f \in V^* \mid \forall i \quad f(\vec{v}_i) \geq 0 \}$$

For each i

$\{ f \in V^* \mid f(\vec{v}_i) \geq 0 \}$ is a half-space through $\vec{0}$.

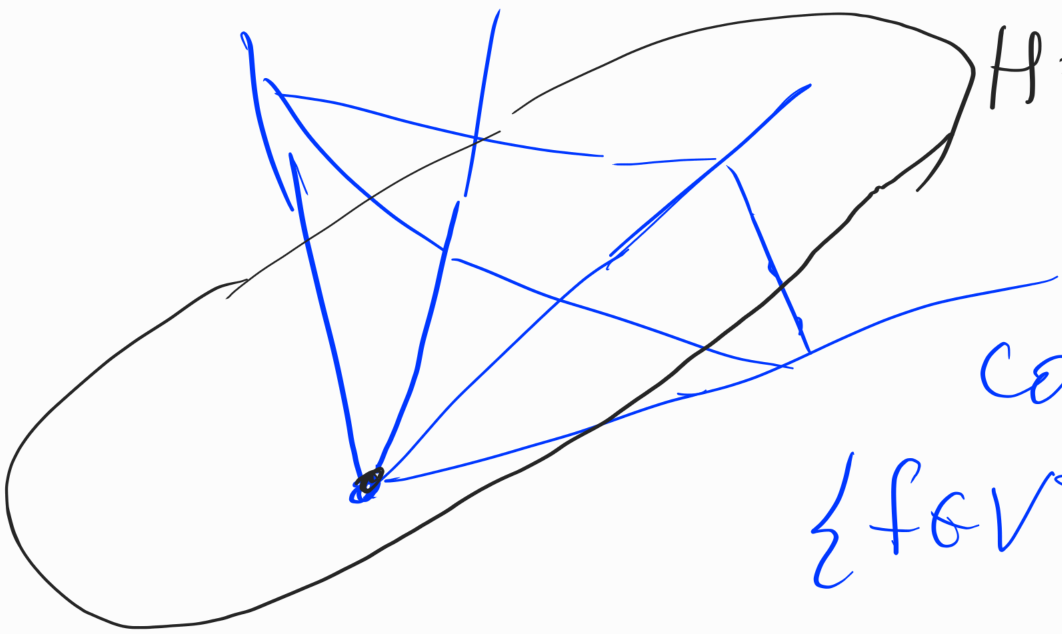
For one such cone, it is polyhedral. (Base of induction for $N = 1$).

<u>Step</u>. Suppose it is true

that

$$\bigcap_{i=1}^{N-1} \{f \in V^* \mid f(\vec{v}_i) \geq 0\} = C$$

is a polyhedral cone.



$H = \{f \in V^* \mid f(\vec{v}_N) = 0\}$

$\parallel$

cospan

$\{f \in V^* \mid f(\vec{v}_N) \geq 0\}$

<u>Claim.</u> $C \cap H$ is a polyhedral cone.

(By Induction Hypothesis)

So $\exists\, \vec{g}_1, \dots, \vec{g}_K \in C \cap H$

s.t. $C \cap H = \langle \vec{g}_1, \dots, \vec{g}_r \rangle$

We also have generators

of C: $\vec{c}_1, \dots, \vec{c}_m$

If $\vec{c}_i(\vec{v}_N) \geq 0$, we

keep it: $\vec{c}_i \in C \cap \{f \mid f(\vec{v}_N) \geq 0\}$

Take any $\vec{c} \in C$

s.t. $c(\vec{v}_N \geq 0)$

$\vec{c} = \sum a_i \vec{c}_i \qquad a_{i_N} \geq 0.$

Suppose that $\vec{c}_1$ is such

that $c_1(\vec{v}_N) < 0.$

$\vec{c} + t \cdot \vec{c}_1$

$t \in \mathbb{R}_{\geq 0}$ As $t \nearrow$

$$\vec{c} + t\vec{c_1} = (a_1 + t)\vec{c_1} + \ldots$$

$$\begin{cases} a_1 + t = 0 \\ (\vec{c} + t\vec{c_1})(\vec{v_N}) = 0 \end{cases}$$