Lemma. Intersection of a polyhedral cone with a half-space is a polyhedral cone.

$C = \langle c_1, \dots c_m \rangle$

$H = \{ f = 0 \}$

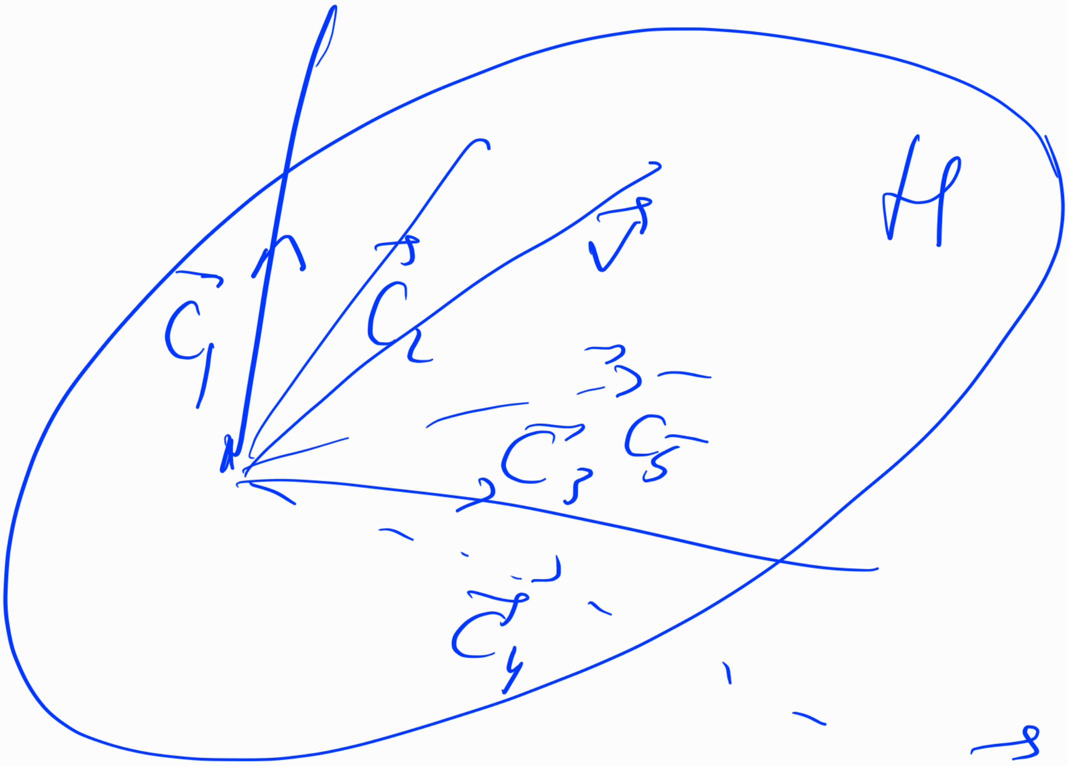
$C' = C \cap \{ f \geq 0 \}$

$C \cap H$ is polyhedral

$C \cap H = \langle g_1 \dots g_k \rangle$

Claim. Those generators $\vec{c_i}$ that $f(c_i) \geq 0$ and all $g_1 \dots g_k$

generate $C \cap \{f \geq 0\}$

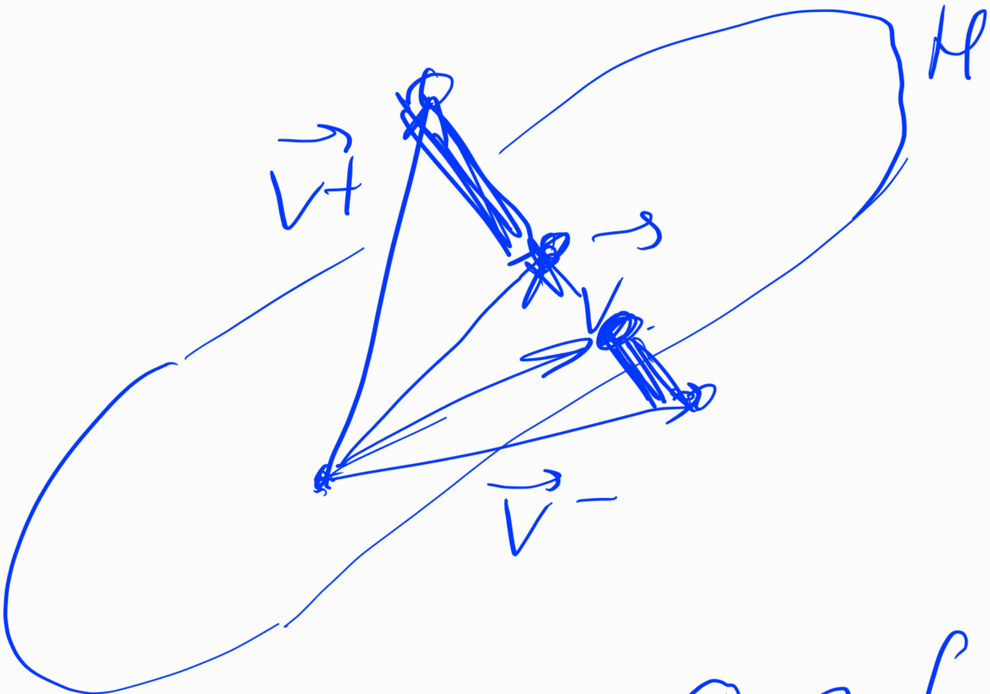


$\vec{v} \in C \qquad \vec{v} = \sum a_i \vec{c_i}$

$$\vec{v} = \frac{\vec{v}^+ + \vec{v}^-}{2}$$

$$\vec{v}^+ = 2 \sum_{f(\vec{c_i}) \geq 0} a_i \vec{c_i}$$

$$\vec{v}^- = 2 \sum_{f(\vec{c_i}) < 0} a_i \vec{c_i}$$



Suppose $\vec{v} \in C \cap \{f \geq 0\}$

$$\vec{v}_0 = \sum b_j \vec{g}_j$$

$$\vec{v} = \alpha \cdot \vec{v}^+ + (1-\alpha)\vec{v}_0$$

where $\alpha \in [0,1]$

so $\vec{v}$ is a non-negative linear combination of $\vec{c}_i$ with $f(c_i) \geq 0$ and g_j $\square$.

Moreover If the original cone σ is rational polyhedral, then so is $\hat{\sigma}$.

Construction of a general toric variety from a fan

Recall

$$M = \mathbb{Z}^n \subseteq \mathbb{R}^n$$

"lattice of monomials"

$$M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$$

$N = \mathrm{Hom}(M, \mathbb{Z})$

"lattice of valuations"

or "lattice of 1-dimensional subgroups"

<u>Affine toric variety</u>

$\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ $\quad N_{\mathbb{R}}$

rational, polyhedral, closed, strictly convex

(cospan $= \{0\}$)

$X_\sigma = \mathrm{mSpec}(\mathbb{C}[\ \hat{\ } \cap M])$

σ strictly convex

$\hat{\sigma}$ is of maximal dimension.

$\mathrm{span}(\hat{\sigma}) = M_{\mathbb{R}}$

Ex.

$\sigma = \{\vec{0}\}$ $\hat{\sigma} = M_{\mathbb{R}}$

$\hat{\sigma} \cap M = M$

$M = \mathbb{Z}^n \ni \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$\updownarrow$

$x_1 \ldots x_n$ $\underline{x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}}$

"monic Laurent monomial"

$\mathbb{C}[M] = \mathbb{C}[x_1, \ldots, x_n, \frac{1}{x_1}, \ldots \frac{1}{x_n}]$

Laurent polynomials in n variables.

$m\mathrm{Spec}(\mathbb{C}[M]) \cong (\mathbb{C}^*)^n$

Def. A fan in N is a ~~fin~~ finite collection of strictly convex closed rational polyhedral cones $\{\sigma\}$ in N s.t.

1) Every face of each σ in the fan is in the fan

2) For all σ_1, σ_2 in the fan $\sigma_1 \cap \sigma_2$ is a face in σ_1 and a face in σ_2

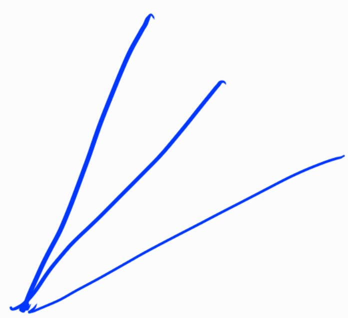
Def. σ is a polhedral cone, f is a linear functional s.t. $f(\vec{v}) \geq 0$ $\forall \vec{v} \in \sigma$

Then the cone

$\sigma \cap \{f=0\}$

is called a face of σ.

Ex. If $\sigma = \langle \vec{c}_1, \ldots, \vec{c}_k \rangle$



$\{\vec{c}_i\}$ are linearly independent

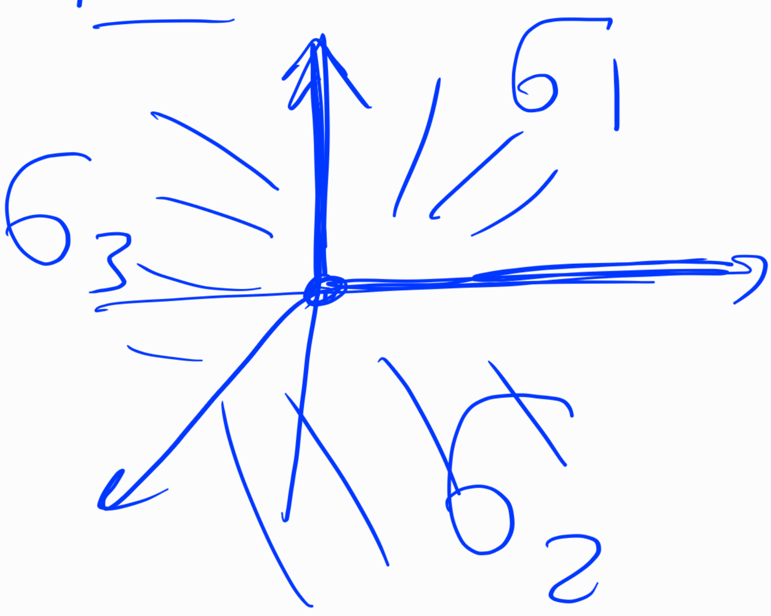
Then the faces are

$\langle c_i \mid i \in S \rangle$

where S is any subset of $\{1, 2, \ldots, k\}$

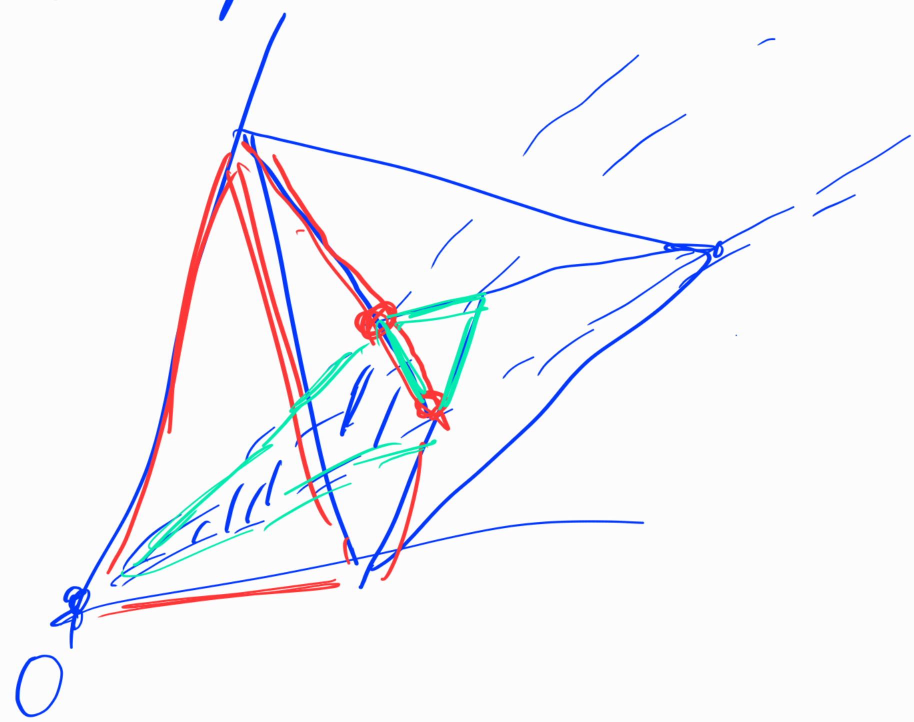
Def. A facet is a face of codimension 1.

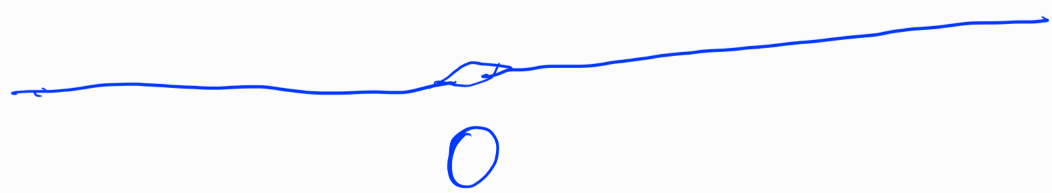
Ex. $n=2$



Fan: $\{\sigma_1, \sigma_2, \sigma_3$
$\sigma_1 \cap \sigma_2, \sigma_1 \cap \sigma_3, \sigma_2 \cap \sigma_3, \{0\}\}$

Wikipedia "Toric varieties"





Main construction:

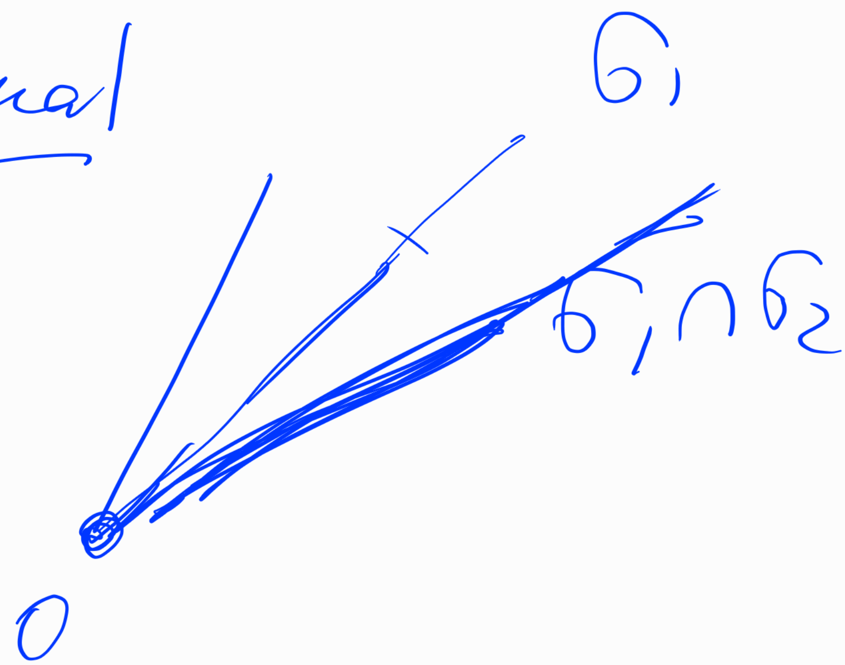
we glue X_{σ_1} and X_{σ_2}

by $X_{\sigma_1 \cap \sigma_2}$

$\sigma_1 \cap \sigma_2$ is a <u>face</u> in σ_1: it is given by imposing $f(\vec{v}) =$ when $f \in \hat{\sigma}_1$

we can choose f to be <u>rational</u>

moreover <u>integer</u>



$f \in \hat{\sigma}_1 \cap M$

<u>Claim.</u>

$\mathbb{C}[\widehat{\sigma_1 \cap \sigma_2 \cap M}]$

is the locali of $\mathbb{C}[\hat{\sigma_1}]$ w.r.t.

$\{1, f, f^2, \dots\}$

(identify f with the corresponding monomial)

Specifically:

all $g \in \widehat{\sigma_1 \cap \sigma_2 \cap M}$

can be written as

$g_1 - k \cdot f$, $\quad k \geq 0$

$g_1 \in \hat{\sigma_1}$

Ex.