ON THE STEIN FACTORIZATION OF RESOLUTIONS OF JACOBIAN SELF-MAPS OF $\mathbb{P}^2$

ALEXANDER BORISOV

Abstract. We study the resolutions of rational self-maps of the projective plane that come from hypothetical counterexamples to the two-dimensional Jacobian Conjecture and establish several strong restrictions on its structure, in particular, its Stein factorization. We prove that all curves at infinity there go through a single point, and that the di-critical log-ramification divisor is ample. As a simple corollary, we prove that the Stein factorization of a counterexample to the JC must be singular.

1. Introduction

Suppose $f(x,y)$ and $g(x,y)$ are two polynomials with complex coefficients. The classical Jacobian Conjecture (due to Keller, [8]) asserts the following.

Conjecture. (Jacobian Conjecture in dimension two) If the Jacobian of the pair (f,g) is a non-zero constant, then the map $(x,y) \mapsto (f(x,y), g(x,y))$ is invertible. Note that the opposite is clearly true, because the Jacobian of any polynomial map is a polynomial, and, when the map is invertible, it must have no zeroes, so it is a constant.

This paper initiates an approach to this conjecture, that the author considers to be the most natural from the point of view of modern biratonal geometry. It should be noted that many excellent mathematicians attempted to solve the Jacobian conjecture, including a number of birational geometers, beginning with Keller himself. See an excellent survey of Miyanishi [10] for some references. Fro the more algebraic approaches see the survey of van den Essen [7]. The term "Jacobian Conjecture" was coined by Abhyakar (cf. [1]).

This paper is written primarily for the algebraic geometers, by an algebraic geometer. So we take for granted the properties of canonical divisors on normal surfaces, while working out carefully some elementary combinatorial results. All varieties in this paper are over complex numbers, even though many results hold true in broader generality.

The research of the author was supported in part by the NSA grants H98230-08-1-0129, H98230-06-1-0034 and H98230-11-1-0148.

From the point of view of a birational geometer, the most natural approach to the two-dimensional Jacobian Conjecture is the following. Suppose a counterexample exists. It gives a rational map from $X = \mathbb{P}^2$ to $Y = \mathbb{P}^2$. After a sequence of blow-ups of points, we can get a surface Z with two morphisms: $\pi : Z \to X$ (projection onto the origin P^2) and $\varphi : Z \to Y$. (the lift of the original rational map).

Note that Z contains a Zariski open subset isomorphic to $\mathbb{A}^2$ and its complement, $\pi^*((\infty))$, is a tree of smooth rational curves. We will call these curves exceptional, or curves at infinity. The structure of this tree is easy to understand inductively, as it is built from a single curve (∞) on $\mathbb{P}^2$ by a sequence of two operations: blowing up a point on one of the curves or blowing up a point of intersection of two curves. However, a non-inductive description is probably impossible, which is the first difficulty in this approach. Another difficulty comes from the fact that the exceptional curves on Z may behave very differently with respect to the map φ. More precisely, there are four types of curves E.

type 1) $\varphi(E) = (\infty)$

type 2) $\varphi(E)$ is a point on (∞)

type 3) $\varphi(E)$ is a curve, different from (∞)

type 4) $\varphi(E)$ is a point not on (∞)

From a first glance, the situation appears almost hopeless. The curves of type 3 are especially cubersome, they are known as di-critical components (cf., e.g. [11], [2]). One of the goals of this paper is to bring new life to this naive approach by showing that this a priori intractable collection of data has some very rigid structure. In particular, for a given graph of curves on Z, one can essentially always tell which curves are of which type, and there is a fairly restrictive family of graphs that can potentially appear in a counterexample to the Jacobian Conjecture. Our main tools are the basic tools of algebraic geometry of surfaces: the intersection pairing and the adjunction formula. We are guided by some of the ideas of the log Minimal Model Program, but most of the proofs are relatively elementary and self-contained. This paper is a beginning of a bigger investigaton, see the preprint [4] for further developments.

Our first new tool is not exactly new: we label the exceptional curves by the coefficients of the log canonical class in the basis of exceptional curves. We use this labeling in Section 2 to obtain some preliminary results on the graph of the exceptional curves on Z. In section 3 we use slightly more subtle arguments, with inequalities in the spirit of the paper [3], to study the Stein factorization of the morphism φ. Our main result is the following theorem, which is a combination of Theorems 3.4 through 3.8.

Theorem. Suppose $\tau : Z \longrightarrow W, \rho : W \longrightarrow Y$ is the Stein factorization of φ, for a counterexample to the Jacobian Conjecture. Suppose E_i are images of the exceptional curves on W (the curves of types 1 and 3). Then the following are true.

1) $W \setminus (\bigcup_i E_i)$ is isomorphic to the affine plane.

2) All curves E_i pass through the point $A = \tau(\pi^{-1}(\infty))$, where (∞) is the line at infinity on X, and have no other points of intersection.

3) All curves E_i of type 1 are smooth outside of A.

4) Each curve E_i of type 3 is either smooth outside of A or contains exactly one point $A_i \neq A$ so that it is smooth outside of A and A_i.

5) The surface W is singular at A and is nonsingular outside of A and A_i. Each A_i is (complex analytically) a cyclic quotient singularity.

6) For each E_i denote by ρ_i the map between normalizations of E_i and $\rho(E_i)$, induced by ρ. Then for each E_i that does not contain A_i, the map ρ_i is an isomorphism. For each curve E_i that contains A_i, $E_i \setminus \{A, A_i\}$ is isomorphic to the algebraic torus (i.e. $Spec(\mathbb{C}[x, x^{-1}])$) and ρ_i is isomorphic to a map $x \mapsto x^{f_i}$ for some $f_i \in \mathbb{N}$.

7) Suppose r_i is the ramification index of ρ at E_i. Denote by $\bar{R} = \sum_{type(E_i)=3} r_i E_i$ the di-critical log-ramification divisor on W. Then $\bar{R}$ is ample.

Acknowledgments. This paper is dedicated to the memory of V.A. Iskovskikh, who introduced the author to the beauty of birational geometry. The author is also indebted to David Wright and Ed Formanek for stimulating discussions related to the Jacobian Conjecture.

2. Preliminary Observations and Definitions

We follow the notation from the Introduction.

Suppose $X = \mathbb{P}^2$, $Y = \mathbb{P}^2$ and $\varphi_Y^X : X - -- > Y$ is a rational map. Suppose further that on an open subset $\mathbb{A}^2 \subset \mathbb{P}^2 = X$ the map φ_Y^X is defined, unramified, and $\varphi_Y^X(\mathbb{A}^2) \subseteq \mathbb{A}^2 \subset \mathbb{P}^2 = Y$. By a sequence of blow-ups at smooth points, we get a surface Z with a birational map $\pi : Z \to X$ and a generically finite map $\varphi_Y^Z : Z \to Y$ such that $\varphi_Y^Z = \varphi_Y^X \circ \pi$. We will denote $\varphi = \varphi_Y^Z$.

The blow-ups that lead to Z can be done outside of $\mathbb{A}^2 \subset X$. So $Z = \mathbb{A}^2 \cup (\cup E_i)$, where E_i are rational curves. The following proposition collects some straightforward observations.

Proposition 2.1. *1) The curves E_i form a tree.*

2) One of E_i is $\pi^{-1}(\infty)$, all others are mapped to points by π.

3) The classes of E_i form a basis of the Picard group of Z.

The structure of Z is to a large extent determined by the graph of intersections of E_i. The vertices of this graph correspond to E_i-s and are usually labeled by E_i^2. The edges correspond to the points of intersections of two different E_i-s. The graph is a tree.

This graph is not so easy to deal with because blowing up a point changes the self-intersections of the curves passing through it. Inspired by the Minimal Model Program, we consider a different labeling of this graph. We consider the augmented canonical class of Z, $\bar{K}_Z = K_Z + \sum_i E_i$. It can be uniquely written as a linear combination of E_i, $\bar{K}_Z = \sum_i a_i E_i$. We label the vertices of the intersection graph by these numbers a_i.

With this labeling we now describe what happens when a point is blown up, in any of the intermediate steps in getting from X to Z.

Proposition 2.2. *When a point is blown up, going from Z' to Z'', one of the following two operations is performed to the graph of the exceptional curves:*

1) A new vertex is added to the graph, connected to one of the vertices. It is labeled a_i+1, where a_i is the label of the vertex it is connected to.

2) A new vertex is introduced on the edge connecting two vertices, "breaking" the edge into two edges. The new vertex gets labeled with $a_i + a_j$, where a_i and a_j are the labels of the two vertices it is connected to.

Proof. The first case corresponds to blowing up a point on one of the curves. The second case corresponds to blowing up an intersection of two curves. The augmented canonical class calculations are straightforward and are left to the reader. $\square$

Notice that once a vertex is created, its label never changes, which is in sharp contrast with the traditional labeling.

The following observation is true for any Z, unrelated to the map φ. It is easily proven by induction on the number of exceptional curves, using the above proposition.

Proposition 2.3. *For any two adjacent vertices E_i, E_j of the graph of Z, $gcd(a_i, a_j) = 1$. In particular, no two adjacent vertices have even labels.*

For most of the exceptional curves, one can easily recover their self-intersection from this $\bar{K}$-labeled graph, using the adjunction formula:

$(K_Z + E_i)E_i = -2$, so $\bar{K}_Z \cdot E_i = -2 + \#(E_j \text{ adjacent to } E_i)$

Thus, if $\bar{K}_Z = \sum a_i E_i$, we have

$$a_i E_i^2 + \sum_{E_j \ adj. \ E_i} a_j = -2 + \#(E_j \text{ adjacent to } E_i)$$

So if $a_i \neq 0$, E_i^2 can be easily calculated.

However, when $a_i = 0$, it is not that easy. In fact, the graph in the example below can be obtained from the graph of $\mathbb{P}^2$ by a sequence of blowups described above. In this symmetric graph, $\pi^{-1}(\infty)$ is the third curve from the left; the self-intersection of the left curve is (-1), while the self-intersection of the right curve is (-2). See [4] for the detailed construction.

Example

$$\begin{array}{ccccccccccccc} \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ 0 & & -1 & & -2 & & -1 & & -2 & & -1 & & 0 \end{array}$$

Note that the subgraph of vertices with negative labels is connected. It is separated from the "positive" vertices by the "zero" vertices. Moreover, the "zero" vertices are only connected to vertices with labels (-1) or 1.

Now we are going to make use of the map φ. The main idea is to use the adjunction formula for φ to get a formula for $\bar{K}_Z$.

Recall from the Introduction the four types of curves E_i. For every curve of type 1 or 3 denote by f_i the degree of the map onto its image and by r_i the ramification index. Denote by L the class of the line on $Y = \mathbb{P}^2$.

Proposition 2.4. *There exist integers b_i for the curves E_i of types 2 and 4 such that*

$$\bar{K}_Z = \varphi^*(-2L) + \sum_{type(E_i)=3} r_i E_i + \sum_{type(E_i)=2or4} b_i E_i$$

Proof. Consider the differential 2-form ω on $Y = \mathbb{P}^2$ that has the pole of order 3 at (∞) and no other poles or zeroes. Because φ is unramified on the $\mathbb{A}^2 \subset X$, there is a differential form on Z, such that its divisor of zeroes and poles is $\varphi^*(-3L) + \sum_i c_i E_i$, where c_i can be calculated locally at a general point of each E_i.

Notice that for the curves E_i of types 1 and 3, $c_i = r_i - 1$, and

$$\varphi^*(L) = \sum_{type(E_i)=1} r_i E_i + \sum_{type(E_i)=2} e_i E_i$$

for some e_i. Thus,

$$\bar{K}_Z = K_Z + \sum E_i = \varphi^*(-3L) + \sum_{type(E_i)=1or3} r_i E_i + \sum_{type(E_i)=2or4} (c_i+1) E_i =$$

$$= \varphi^*(-2L) + \sum_{type(E_i)=3} r_i E_i + \sum_{type(E_i)=2or4} b_i E_i$$

□

Note that because E_i are independent in the Picard group of Z, the above representation of $\bar{K}_Z$ is unique and must match with the labeling of the graph of E_i. As a corollary, we have the following observation.

Proposition 2.5. *1) Any curve of type* 1 *has a negative even label.*
2) Any curve of type 3 *has a positive label.*

Proof. Note that $\varphi^*(-2L)$ only involves curves of type 1 and 2. $\square$

Additionally, the union of curves of type 1 and 2 must be connected, as a specialization (set-theoretically) of a pullback of a generic L on $Y = \mathbb{P}^2$. This means that the corresponding subgraph is connected.

Every curve of type 3 must intersect with one of the curves of type 1 or 2, while the curves of type 4 do not intersect with curves of type 1 or 2. (This follows from the projection formula of the intersection theory: if E is a curve on Z, $E \cdot \varphi^*(L) = (\varphi_* E) \cdot L$.)

On the other hand, a type 3 curve cannot intersect a type 1 curve, because negative and positive labels are never adjacent. Because the graph of the exceptional curves on Z is a tree, no two curves of type 3 intersect with each other. Putting this all together, we must have the following. The tree of curves on Z has a connected subtree containing all curves of type 1 and 2. Some of the vertices of this subtree may have one or more curves of type 3 connected to them. Then some of these type 3 curves may have trees of type 4 curves connected to them. Additionally, no two curves of type 1 are adjacent, and the subtree of curves of type 1 and 2 contains the connected subtree of curves with negative labels.

Proposition 2.6. $\pi^{-1}(\infty)$ *is of type* 1 *or* 2.

Proof. One can prove it using the above description of the graph of exceptional curves, but there is also the following direct geometric argument. The pullbacks of lines on $X = \mathbb{P}^2$ form a family of rational curves C on Z that intersect $\pi^{-1}(\infty)$ at a generic point and do not intersect any other exceptional curves. Consider $\varphi(C)$ for a generic C. If $\pi^{-1}(\infty)$ is of type 3 or 4 then $\varphi(C) \subseteq \mathbb{A}^2 \subset Y$. The curve C is proper and $\mathbb{A}^2$ is affine, so $\varphi(C)$ is a point, which is impossible. $\square$

Until now, the variety Z was an arbitrary resolution at infinity of the original rational map. But we can put an additional restriction on it, to avoid unnecessary blow-ups. In what follows we abuse the terminology slightly by identifying curves on birationally equivalent surfaces, that correspond to the same divisorial valuation.

Definition 2.1. *If a curve is obtained by blowing up the intersection of two curves, we call these curves its parents. If a curve is obtained by* *blowing up a point on one of the curves, this curve is called its parent. The original line at infinity has no parents. Note that other curves may be created afterwards that separate the curve from one or both of its parents.*

Definition 2.2. *For a given curve E, the set of its ancestors $A(E)$ is the smallest set S of the exceptional curves that contains its parent(s) and has the property that it contains the parents of every curve in S. Note that this set is empty if E is the original line at infinity. Otherwise, it consists of the original line at infinity and all curves that have to be created before E.*

Definition 2.3. *A curve E_i on Z is called* **final** *if there is a sequence of blow-ups from X to Z such that E_i is blown up last. Equivalently, a curve is final if it is not a parent to any exceptional curve on Z.*

Note that there may be more than one final curve, and $\pi^{-1}(\infty)$ is never final. In what follows, E_i is one of the exceptional curves on Z.

Proposition 2.7. *Suppose that when Z was created, E_i was created after all of its neighbors in the graph (i.e. all adjacent vertices). Then E_i is a final curve.*

Proof. Instead of creating E_i at its due time, we can change the order of blow-ups and create it at the last step of the process, without changing anything else. □

Proposition 2.8. *Suppose $a_i = a(E_i) \geq 2$ and it is the largest label among all its neighbors. Then E_i is final.*

Proof. We will prove that E_i was created after all its neighbors. First of all, no neighbor of E_i can be a blow-up of a point on E_i, because its label would have been $a_i + 1$. If it were a blow-up of a point of intersection of E_i and some E_j, then E_i and E_j were adjacent before the blow-up. Negative curves are never adjacent to the positive curves and zero curves are only adjacent to curves with labels 1 of -1. Thus, $a_j \geq 1$. So the label of the new curve is $a_i + a_j \geq a_i + 1 > a_i$. □

Note that no two curves with the same label $a_i \geq 2$ can be adjacent, by Proposition 3. So every local maximum $a_i \geq 2$ is strict.

Proposition 2.9. *If $a_i = 1$, then E_i is final if and only if it either has only one neighbor, with label 0, or exactly two neighbors, with labels 1 and 0.*

Proof. A curve with label 1 can be created either by a blow-up of a point on a curve with label 0 or by a blow-up of an intersection of a curve with label 0 and a curve with label 1. Once created, it will be final if and only if no other curve is blown up as its neighbor. The rest is easy and is left to the reader. □

The above two propositions allow us to easily spot the final curves in the positive part of the graph of curves. Our interest in the final curves stems from the following. If one of the final curves on Z is of type 2 or 4, then it can be contracted, using the $\varphi-$relative MMP, to get another Z, with two maps to X and Y and a smaller Picard number.

Definition 2.4. *We call Z* **minimal** *if all of its final curves are of type* 1 *or* 3*.*

Proposition 2.10. *If a counterexample to the Jacobian Conjecture exists, it can be obtained using a minimal Z.*

Proof. Take Z with smallest possible Picard number. If it is not minimal, it can be created in such a way so that some curve of type 2 or 4 is blown-up last. Using MMP relative to φ, it can be blown down, maintaining the morphisms, and creating a counterexample to the Jacobian Conjecture with smaller Picard number. □

From now on, Z will always be minimal.

Proposition 2.11. *Suppose E is a curve of type* 3 *on Z. Suppose E_0 is the curve of type* 2 *it is adjacent to. Then the tree on the other side of E is a line $E - E_1 - ... - E_k$, where $E_1, ...E_k$ are of type* 4*.*

Proof. The label of E is positive. All curves $E_1, ...E_k$ "on the other side" of E are of type 4. They must be ancestors of some curve of type 3, so they are all ancestors of E. If the connected component of the graph obtained from Γ by removing E is not a line, there would have to be another final curve curve there, which is impossible. □

3. Other Varieties and Further Analysis

We start with the theorem that shows that type 3 curves must exist in a counterexample to the Jacobian Conjecture. Note that the type 3 curves are called "di-critical components" in [5], [6], and this fact is well known and can be easily proven by a topological argument. So the main purpose of our proof is to show an easy application of our method before proceeding to the more intricate questions.

Theorem 3.1. *Suppose Z and φ provide a counterexample to the Jacobian Conjecture. Then Z contains a curve of type* 3*, where φ is ramified.*

Proof. Consider a generic line L on the target variety $Y = \mathbb{P}^2$. The curve $C = \varphi^{-1}(L)$ is smooth and irreducible ("Bertini's theorem").

Moreover, we can assume that for all but finitely many lines L' that only intersect L "at infinity", $C' = \varphi^{-1}(L')$ is smooth and irreducible. We can also assume that L does not pass through the images of the exceptional curves of types 2 and 4, so C does not intersect these curves on Z. Suppose that the genus of C is g, the map $H = \varphi_{|C} C \to L$ has degree n and the number of points of C "at infinity" is k. (There is a special point ∞ on L, the only one not lying in $\mathbb{A}^2$. The number k is the number of points of C mapped to it, in a set-theoretic sense.) Because the map φ is only ramified at the exceptional curves of X, the map H could only by ramified at these k points at infinity. By Hurwitz formula, we have

$$2g - 2 = -2n + r,$$

where r is the total ramification at infinity. We have $g \geq 0$, $n \geq 1$ and $r \leq n - k$. So

$$-2 \leq 2g - 2 \leq -2n + n - k = -n - k \leq -2$$

Thus all the inequalities above are equalities, $g = 0$, $n = 1$, and $k = 1$. Because $n = 1$, the map φ is birational. For the birational maps the Jacobian Conjecture is well known (see, e.g. [1]). $\square$

Now we want to make further use of the morphism $\varphi : Z \to Y$. We decompose it into a composition of two morphisms, birational and finite (Stein factorization):

$$Z \longrightarrow W \longrightarrow Y$$

Here the first morphism is birational and denoted τ, and the second one is finite and denoted ρ.

The surface W is normal. In what follows, we use the intersection theory for normal surfaces due to Mumford. Suppose K_W is its canonical class, as the Weil divisor class modulo numerical equivalence. We define the augmented canonical class $\bar{K}_W = K_W + \sum E_i$.

Proposition 3.1. *In the situation and notation described above,*

$$\bar{K}_W = \rho^*(-2L) + \sum_{type(E_i)=3} r_i E_i.$$

Proof. The curves E_i on W are exactly the images of curves of types 1 and 3 on Z. By adjunction, we have:

$$K_W = \rho^* K_Y + \sum (r_i - 1) E_i,$$

where r_i is the ramification index, and E_i are the images of the curves E_i of types 1 and 3 on Z.

$$\bar{K}_W = \rho^*(-3L) + \sum r_i E_i = \rho^*(-2L) + \sum_{type(E_i)=3} r_i E_i.$$

□

Note that if one denotes by $\bar{K}_Y$ the class of $K_Y + (\infty)$, then $\rho^*(-2L)$ in the above Proposition is $\rho^*(\bar{K}_Y)$. The next theorem is very important. It will be further strengthened in Theorem 3.7.

Theorem 3.2. *(Big Ramification Theorem)*
Suppose Z is a counterexample to the Jacobian Conjecture. Then on W the "di-critical log-ramification divisor"

$$\bar{R} = \sum_{E_i \subset W, type(E_i)=3} r_i E_i$$

intersects positively with all exceptional curves of type 3*. As a corollary, $\bar{R}^2 > 0$.*

Proof. Suppose that $C = E_i$ is a curve of type 3 on W. Suppose d_i is the degree of $\rho(C)$. Suppose $\tau : Y_2 \to W$ is a minimal resolution of singularities of W. Then

$$\bar{K}_W E_i = (K_W + E_i)E_i + \sum_{j \neq i} E_i E_j,$$

where E_j are curves of type 1 or 3. Note that because E_i intersects at least one curve of type 1, $\bar{K}_W E_i > (K_W + E_i)E_i$. Lifting up to Y_2, we get

$$(K_W + E_i)E_i = (\tau^*(K_W) + \tau^*(E_i))\tau^{-1}(E_i) \geq (K_{Y_2} + \tau^{-1}(E_i))\tau^{-1}(E_i) \geq -2.$$

So for all i $\bar{K}_W E_i > -2$.

Therefore,

$$\bar{R} \cdot E_i = 2\varphi^*(L) \cdot E_i + \bar{K}_W \cdot E_i > 2 f_i d_i - 2 \geq 0$$

□

Corollary 3.1. *The curve $\pi^{-1}(\infty)$ is of type* 2*.*

Proof. By Proposition 2.6, it is of type 1 or 2. If it is of type 1, then it is not included in $\sum_{type(E_i)=3} r_i \tau^* E_i$. So $\sum_{type(E_i)=3} r_i \tau^* E_i$ consists of curves contractible by π. But

$$(\sum_{type(E_i)=3} r_i \tau^* E_i)^2 = (\sum_{type(E_i)=3} r_i E_i)^2 > 0,$$

contradiction. $\square$

Note that every curve of type 3 on Z intersects the union of curves of type 2 at exactly one point, and does not intersect curves of type 1. When the curves of type 2 are contracted, on W, every curve of type 3 intersects the union of curves of type 1 at exactly one point.

Proposition 3.2. *For every curve E_i of type 3 on W the point above is $\tau(\pi^{-1}(\infty))$.*

Proof. Suppose there is a point $w \in W$ on the union of type 1 curves, which is not $\tau(\pi^*(\infty))$ and which has some type 3 curves passing through it. Define

$$\bar{R}_w = \sum_{w \in E_i, type(E_i)=3} r_i E_i$$

Because the curves of type 3 not passing through y cannot intersect any components of $\bar{R}_w$, we have $\bar{R}_w^2 = \bar{R}_w \cdot \bar{R}$. By Theorem 3.2, this implies that $\bar{R}_w^2 > 0$. Like in the proof of Corollary 3.1, $\tau^*(\bar{R}_w)$ consists of curves contractible by π, which is impossible. $\square$

Proposition 3.3. *On W, all exceptional curves contain $\tau(\pi^{-1}(\infty))$ and there are no other points of intersection.*

Proof. By the proposition above, every curve of type 3 contains $\tau(\pi^{-1}(\infty))$ and this is its only point of intersection with other exceptional curves. Now consider a curve E_i of type 1. Suppose it does not contain $\tau(\pi^{-1}(\infty))$. Then it does not intersect any of the curves of type 3 on W.

On W we have:

$$\bar{K}_W \cdot E_i \geq (K_W + E_i)E_i \geq -2$$

On the other hand,

$$\bar{K}_W \cdot E_i = (-2\rho^*(L) + \bar{R}) \cdot E_i = -2\rho^*(L) \cdot E_i \leq -2$$

The inequalities above become equalities only if E_i intersects no other curves and is smooth. This would make it the only curve of type 1 on W, which would then have to intersect with some curves of type 3, contradiction. $\square$

Thus, we know that every curve of type 1 on W contains $\tau(\pi^{-1}(\infty))$. We now look at the graph of curves on Z. The curves of type 2 that are mapped to $\tau(\pi^{-1}(\infty))$ form a connected subgraph, containing $\pi^{-1}(\infty)$. Every curve of type 1 or 3 is attached to this subgraph. On "the other side" of each curve of type 3 there may be a single chain of curves of type 4, and on "the other side" of each curve of type 1 there may be a single chain of curves of type 2. Note that all of these "other side" curves must be created before the corresponding type 3 or type 1 curves. When mapped to W, the curves of type 1 and 3 intersect at $\tau(\pi^{-1}(\infty))$ and nowhere else.

One can restrict the structure of the possible counterexamples even further.

Theorem 3.3. *In any counterexample to the Jacobian Conjecture there are no curves on "the other side" of the curves of type* 1*.*

Proof. Consider a curve of type 1, E, on Z. Suppose the ramification index at E is r. Then the coefficient of $\varphi^*(L)$ in E is r, and the coefficient of $\bar{K_Z}$ is $(-2r)$. Consider the divisor class $D = \bar{K_Z} + 2\varphi*(L) = ... + 0 \cdot E + x_1E_1 + ... + x_kE_k$, where $E_1, ...E_k$ are the curves on Z "on the other side" of E. We know that D intersects by zero with $E_1, ..., E_{k-1}$. It intersects by -1 with E_k. We formally add another vertex to the graph, "E_{k+1}" and set the coefficient of D at it to be 1. (Note that we are not blowing up any points and E_{k+1} does not have any geometric meaning). We now have a chain $E, E_1, ..., E_k, E_{k+1}$ and a divisor $D' = 0 \cdot E + x_1E_1 + ... + x_kE_k + 1 \cdot E_{k+1}$, such that D' intersects by zero with all $E_1, E_2, ..., E_k$. Because the self-intersections of all E_i, $1 \leq i \leq k$, are less than or equal to -2, the coefficients x_i must form a concave up chain between 0 and 1, contradicting their integrality. (Here is a more formal argument. Suppose at least one of the x_i, $1 \leq i \leq k$, is not positive. Then consider the minimum of x_i, obtained at x_j, such that $x_{j+1} > x_j$, where formally $x_0 = 0, x_{k+1} = 1$. Then $D' \cdot E_j \geq x_{j-1} + x_{j+1} + 2x_j > 0$, contradiction. Suppose the maximum of x_i, $1 \leq i \leq k$, is greater than or equal 1 and is obtained at x_j, where $x_{j-1} < x_j$. Then $D' \cdot E_j \leq x_{j-1} + x_{j+1} - 2x_j < 0$, contradiction. Thus all x_i are strictly between 0 and 1, which is impossible because they are integers.) $\square$

As a corollary of these observations, we get a rather detailed description of the structure of W.

Theorem 3.4. *Suppose W is defined as above for a counterexample to the Jacobian Conjecture, E_i are images of the exceptional curves on it. Then $W \setminus \cup_i E_i$ is isomorphic to the affine plane. There exist distinct points $A = \tau(\pi^{-1}(\infty))$ and $A_i \in E_i$ (at most one for each E_i of type* 3*) so that W is smooth outside of them, A is a normal singularity, A_i are cyclic quotient singularities, all the curves E_i pass through A, do not intersect elsewhere and are smooth in the nonsingular part of W. For all exceptional curves E_i that do not contain A_i, $E_i \setminus \{A\}$ is isomorphic to the affine line. For al curves E_i that contain A_i, $E_i \setminus \{A, A_i\}$ is isomorphic to the algebraic torus (affine line with a removed point).*

Proof. Most of the statements have already been proven. To finish the proof, note the following. The map τ from Z to W contracts all curves of types 2 and 4, and no curves of types 1 and 3. The curves of type 2 form a subtree on Z, so they are contracted to one singular point A. Note that all curves of type 1 and 3 on Z intersect this subtree at exactly one point. Some curves of type 3 have one chain of curves of type 4 attached to them, that get contracted into a cyclic quotient singularity. □

Note that the above theorem restricts greatly the restriction of the map φ to the exceptional curves of types 1 and 3.

Theorem 3.5. *1) For all curves E_i of type 1, $f_i = 1$.*

2) For all curves E_i of type 3 either $f_i = 1$ or the restriction to E_i of the map from Z to Y is isomorphic to the composition of a map $(x \mapsto x^{f_i}) : P^1 \to P^1$ and a generically one-to-one map from P^1 to a possibly singular rational curve (the normalization map for $\varphi(E_i)$).

Proof. This follows from the fact that the restriction to E_i of the map from Z to Y can only be ramified at the points of intersections of E_i and other exceptional curves, and the classification of self-maps of the projective line that are ramified at one or two points. □

For the following theorem, we need to introduce additional notation.

Definition 3.1. *Suppose Z, W, Y and $\varphi : W \to Y$ are as above. For a type 3 curve E_i, denote by F_i its image on Y, as a reduced irreducible divisor (a possibly singular rational curve). Then*

$$\rho^*(F_i) = r_i E_i + G_i,$$

where G_i is an effective Weil divisor. Its irreducible components are curves in $\mathbb{A}^2$ that are mapped to F_i. We will call these curves coexceptional.

Theorem 3.6. *Suppose $E_i \subset W$ is a type 3 curve that contains a cyclic quotient singularity A_i. Then some coexceptional curve from G_i contains A_i.*

Proof. The proof is very similar to the proof of Theorem 3.3. Suppose the support of G_i does not contain A_i. To simplify the notation, denote E_i by E; suppose $E_1, E_2, ..., E_k$ are the curves of type 4 that are mapped to A_i, with E_1 intersecting E. Suppose $\varphi : Z \to Y$ is our map. Consider on Z the divisor $D = \bar{K}_Z - \varphi^*(F_i)$. We can write D as a linear combination of exceptional curves and the strict pullbacks of the coexceptional curves. Because the coefficient of E in this linear combination is zero, and the only curves that contribute to the intersection of D with $E_1, ..., E_k$ are $E_1, ...E_k$, we get a linear combination with integer coefficients $x_1E_1 + ... + x_kE_k$ that intersects by zero with $E_1, ..., E_{k-1}$ and by (-1) with E_k. We can now follow verbatim the argument in Theorem 3.3 to get a contradiction. □

The following is a strengthening of Theorem 3.2.

Theorem 3.7. *(Ample Ramification Theorem)*

Suppose Z is a counterexample to the Jacobian Conjecture. Then on W the "di-critical log-ramification divisor"

$$\bar{R} = \sum_{E_i \subset W, type(E_i)=3} r_iE_i$$

is ample.

Proof. The surface W is rational and, therefore, $\mathbb{Q}$-factorial. So the Weil divisor $\bar{R}$ is $\mathbb{Q}$-Cartier. It is effective and by Theorem 3.2 it intersects positively with all of its irreducible components. So by the Nakai-Moishezon criterion it is enough to show that it intersects positively with all irreducible curves C on W that are not the exceptional curves of type 3.

If C is a curve of type 1 on W, then it intersects the support of $\bar{R}$ at $A = \varphi(\pi^{-1}(\infty))$, so $C \cdot \bar{R} > 0$.

If C is any other curve on W, that does not intersect positively with $\bar{R}$, it does not intersect with any curves of type 3 and it must intersect with at least one curve of type 1 (because it cannot be contained entirely in the affine plane). By the Hodge Index Theorem and Theorem 3.2, $C^2 < 0$. Note also that C does not pass through A, so it intersects with at least one curve of type 1 at a smooth point. Because the $\bar{K}$-labels of all curves of type 1 are at most -2, $K_W \cdot C \leq -3$. Therefore, $(K_W + C) \cdot C \leq -3 < -2$, which is impossible.□

We end the paper with an observation that in any counterexample to the JC the surface W must be singular at the point A.

Theorem 3.8. *In the above notation, the surface W must be singular at A. Moreover, for some (possibly equal) curves E_i and E_j of type 1 on W, their intersection $E_i \cdot E_j$ is not an integer.*

Proof. For any E_i of type 1 on W, denote by E_i' the corresponding curve on Z. We can choose Z to be minimal, so E_i' is a final curve, $(E_i')^2 = -1$. Note that in the Picard group of Z we have $E_i' = \tau^*(E_i) - \Delta_i$, where Δ_i is an effective linear combination of curves of type 2. So $(E_i')^2 = E_i^2 + \Delta_i^2$, therefore $E_i^2 > (E_i')^2$. If $E_i^2 \in \mathbb{Z}$, then $E_i^2 \geq 0$.

Now consider $E_i \cdot \rho^*L$. On the one hand, because $f_i = 1$, it equals $L^2 = 1$. On the other hand, it equals $E_i \cdot \sum_{type(E_j)=1} r_j(E_i \cdot E_j)$. If all $E_i \cdot E_j$ are integers, from the argument above $E_i^2 \geq 0$ and for $j \neq i$

$E_i \cdot E_j \geq 1$. So there are at most two curves of type 1 on W. Note additionally that $\sum_{type(E_j)=1} r_j = d$, where d is the degree of the map ρ. If there are two curve of type 1, say E_1 and E_2, then either r_1 or r_2 is at least $d/2$. Since there are no counterexamples to the JC of degree 1 or 2, we get a contradiction for the other E_i. Thus, there is only one curve of type 1 on W, call it E_1. This implies that $r_1 = d$ and $1 = E_1 \cdot \rho^* L = d \cdot E_1^2$. Thus $d = 1$, which is impossible. $\square$

References

[1] S.S. Abhyankar. Lectures on expansion techniques in algebraic geometry. Notes by Balwant Singh. *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*, **57**. Tata Institute of Fundamental Research, Bombay, 1977. iv+168 pp.

[2] S.S. Abhyankar. Dicritical divisors and Jacobian problem. *Indian J. Pure Appl. Math.* **41** (2010), no. 1, 77–97.

[3] A. Borisov. Boundedness theorem for Fano log-threefolds. *J. Algebraic Geom.* **5** (1996), no. 1, 119–133.

[4] A. Borisov. On resolution of compactifications of unramified planar self-maps. Preprint (2012). Web address: http://front.math.ucdavis.edu/1110.5118

[5] A.V. Domrina. Four-sheeted polynomial mappings of C^2. II. The general case. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), no. 1, 3–36; translation in *Izv. Math.* **64** (2000), no. 1, 1–33.

[6] A.V. Domrina, S.Yu. Orevkov. Four-sheeted polynomial mappings of C^2. I. The case of an irreducible ramification curve. (Russian) *Mat. Zametki* **64** (1998), no. 6, 847–862; translation in *Math. Notes* **64** (1998), no. 5-6, 732–744 (1999).

[7] Arno van den Essen. Polynomial automorphisms and the Jacobian conjecture. *Progress in Mathematics*, **190.** Birkhauser Verlag, Basel, 2000.

[8] Ott-Heinrich Keller. Ganze Cremona-Transformationen. *Monatsh. Math. Phys.* **47** (1939), 299306.

[9] W.D. Neumann. On bilinear forms represented by trees. *Bull. Austral. Math. Soc.* **40** (1989), 303–321.

[10] Masayoshi Miyanishi. Recent developments in affine algebraic geometry: from the personal viewpoints of the author. *Affine algebraic geometry, 307378*, Osaka Univ. Press, Osaka, 2007.

[11] S. Yu. Orevkov. On three-sheeted polynomial mappings of C^2. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 6, 1231–1240.

Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA.

E-mail address: borisov@pitt.edu