A Congruence Problem for Polyhedra

Alexander Borisov, Mark Dickinson, and Stuart Hastings

1. INTRODUCTION. We discuss a class of problems about the congruence or similarity of three-dimensional polyhedra. The basic format is the following:

Problem 1.1. Given two polyhedra in $\mathbf{R}^3$ which have the same combinatorial structure (e.g., both are hexahedra with four-sided faces), determine whether a given set of measurements is sufficient to ensure that the polyhedra are congruent or similar.

We will make this more specific by specifying what sorts of measurements will be allowed. For example, in much of the paper, allowed measurements will include distances between pairs of vertices, angles between edges, angles between two intersecting face diagonals (possibly on different faces with a common vertex) or between a face diagonal and an edge, and dihedral angles (that is, angles between two adjoining faces). One motivation for these choices is given below. Sometimes we are more restrictive, for example, allowing only distance measurements.

In two dimensions this was a fundamental question answered by Euclidean geometers, as (we hope) every student who takes geometry in high school learns. If the lengths of the corresponding sides of two triangles are equal, then the triangles are congruent. The SAS, ASA, and AAS theorems are equally well known. The extension to other shapes is not often discussed, but we will have some remarks about the planar problem as well. It is surprising to us that beyond the famous theorem of Cauchy discussed below, we have been unable to find much discussion of the problems we consider in the literature, though we think it is almost certain that they have been studied in the past. We would be appreciative if any reader can point us to relevant results.

Our approach will usually be to look at the problem locally. If the two polyhedra are almost congruent, and agree in a certain set of measurements, are they congruent? At first glance this looks like a basic question in what is known as rigidity theory, but a little thought shows that it is different. In rigidity theory, attention is paid to relative positions of vertices, viewing these as connected by inextensible rods which are hinged at their ends and so can rotate relative to each other, subject to constraints imposed by the overall structure of rods. In our problem there is the additional constraint that in any movement of the vertices, the combinatorial structure of the polyhedron cannot change. In particular, any vertices that were coplanar before the movement must be coplanar after the movement. This feature seems to us to produce an interesting area of study.

Our original motivation for considering this problem came from a very practical question encountered by one of us (SPH). If one attempts to make solid wooden models of interesting polyhedra, using standard woodworking equipment, it is natural to want to check how accurate these models are.[1] As a mathematician one may be attracted first to the Platonic solids, and of these, the simplest to make appears to be the cube. (The regular tetrahedron looks harder, because non-right angles seem harder to cut accurately.)

doi:10.4169/000298910X480081

[1] The usual method of constructing a polyhedron is by folding a paper shell.

It is possible to purchase lengths of wood with a square cross section, called "turning squares" because they are mostly used in lathes. To make a cube, all one has to do is use a saw to cut off the correct length piece from a turning square. Of course, one has to do so in a plane perpendicular to the planes of sides of the turning square. It is obvious that there are several degrees of freedom, meaning several ways to go wrong. The piece cut off could be the wrong length, or you could cut at the wrong angle, or perhaps the cross section wasn't square to begin with. So, you start measuring to see how well you have done.

In this measurement, though, it seems reasonable to make some assumptions. The basic one of interest here is that the saw cuts off a planar slice. You also assume that this was true at the sawmill where the turning square was made. So you assume that you have a hexahedron—a polyhedron with six faces, all of which are quadrilaterals. Do you have a cube? At this point you are not asking a question addressed by standard rigidity theory.

One's first impulse may be to measure all of the edges of the hexahedron, with the thought that if these are equal, then it is indeed a cube. This is quickly seen to be false, because the faces could be rhombi. Another intriguing possibility that we considered early on is that measuring the twelve face diagonals might suffice. However, we found some examples showing that this was not the case, and David Allwright [**2**] gave a short and elegant quaternion-based classification of all hexahedra with equal face diagonals. See also an earlier discussion of this problem in [**13**].[2] Clearly some other configuration of measurements, perhaps including angles, is necessary. It did not take long to come up with several sets of 12 measurements which did the job, but a proof that this number was necessary eluded us.

In our experience most people, even most mathematicians, who are presented with this problem do not see an answer immediately. Apparently the cube is harder than it looks, and so one would like a simpler problem to get some clues. The tetrahedron comes to mind, and so one asks how many measurements are required to establish that a polyhedron with four triangular faces is a regular tetrahedron.

Now we turn to Cauchy's theorem.

Theorem 1.2 (Cauchy, 1839). *Two convex polyhedra with corresponding congruent and similarly situated faces have equal corresponding dihedral angles.*

If we measure the six edges of our triangular-faced object, and find them equal, then we have established congruence of the faces of our object to those of a tetrahedron. Cauchy's theorem tells us that the dihedral angles are the same and this implies the desired congruence.

For a tetrahedron this result is fairly obvious, but for the two other Platonic solids with triangular faces, namely the octahedron and the icosahedron, it is less so. Hence Cauchy's theorem is of practical value to the (extremely finicky) woodworker, and shows that for these objects, the number of edges is at least an upper bound on the number of measurements necessary to prove congruence. From now on we will denote the number of edges of our polyhedron by E, the number of vertices by V, and the number of faces by F. We will only consider simply connected polyhedra, so that Euler's formula, $V + F = E + 2$, holds.

It is not hard to give an example showing the necessity of convexity in Cauchy's result, but it is one where the two polyhedra being compared are, in some sense, far

[2]The face diagonals of a hexahedron are equal when these diagonals form two congruent regular tetrahedra whose edges intersect in pairs. As it turns out, this arrangement is not unique. We have included Allwright's analysis of the possibilities as Appendix A in the posting of this paper to www.arXiv.org.

apart. It was not easy to determine if convexity was necessary for local congruence. Can a nonconvex polyhedron with all faces triangular be perturbed smoothly through a family of noncongruent polyhedra while keeping the lengths of all edges constant? The answer is yes, as was proved in a famous and important paper by R. Connelly in 1979 [**4**].

Cauchy's result also gives us an upper bound for the number of measurements necessary to determine a unit cube: triangulate the cube by dividing each square face into a pair of triangles. Then we have a triangular-faced object with eighteen edges, and by Cauchy's theorem those eighteen edge measurements suffice to determine the cube up to congruence.

However, we can do better. Start by considering a square. We can approach this algebraically by assuming that one vertex of the square is at $(0, 0)$ in the plane. Without loss of generality we can also take one edge along the x-axis, going from $(0, 0)$ to $(x_1, 0)$ for some $x_1 > 0$. The remaining vertices are then (x_2, x_3) and (x_4, x_5) and this leads to the conclusion that to determine five unknowns, we need five equations, and so five measurements. For example, we could measure the four sides of the square and one vertex angle, or we could measure a diagonal instead of the angle.

We then use this to study the cube. Five measurements show that one face is a square of a specific size. Only four more are needed to specify an adjacent face, because of the common edge, and the three more for one of the faces adjoining the first two. The requirement that this is a hexahedron then implies that we have determined the cube completely, with twelve measurements. This is a satisfying result because it shows that E measurements suffice for a cube as well as for the triangular-faced Platonic solids.

However, as remarked earlier, at this stage we have not proved the necessity of twelve measurements, only the sufficiency. One of the most surprising developments for us in this work was that in fact, twelve are not necessary. It is possible to determine a cube (including its size) with nine measurements of distances and face angles. The reason, even more surprisingly, is that only four measurements are needed to determine the congruence of a quadrilateral to a specific square, rather than five as seemed so obvious in the argument above.

We will give the algorithm that determines a square in four measurements in the final section of the paper, which contains a number of remarks about congruence of polygons. For now, we proceed with developing a general method for polyhedra. This method will also handle similarity problems, where the shape of the polyhedron is specified up to a scale change. In determining similarity, only angle measurements are involved. As the reader might expect, in general $E - 1$ measurements suffice, with one additional length required to get congruence.

2. *E* MEASUREMENTS SUFFICE.

In this section we prove that for a *convex* polyhedron P with E edges, there is a set of E measurements that, at least locally, suffices to determine P up to congruence.

We restrict to convex polyhedra mostly for reasons of convenience: many of the results below should be true in greater generality. (One problem with moving beyond convex polyhedra is determining exactly what the term 'polyhedron' should mean: for a recent attempt to give a general definition of the term 'nonconvex polyhedron,' see the beautiful paper [**9**].) To avoid any ambiguity we begin with a precise definition of convex polyhedron.

Definition 2.1. A *closed half-space* is a subset of $\mathbf{R}^3$ of the form $\{ (x, y, z) \in \mathbf{R}^3 \mid ax + by + cz + d \geq 0 \}$ with $(a, b, c) \neq (0, 0, 0)$. A *convex polyhedron* is a subset P

of $\mathbf{R}^3$ which is bounded, does not lie in any plane, and can be expressed as an intersection of finitely many closed half-spaces.

The vertices, edges, and faces of a convex polyhedron P can be defined in terms of intersections of P with suitable closed half-spaces. For example, a *face* of P is a subset of P of the form $P \cap H$, for some closed half-space H, such that $P \cap H$ lies entirely within some plane but is not contained in any line. Edges and vertices can be defined similarly.

The original problem refers to two polyhedra with the same 'combinatorial structure,' so we give a notion of *abstract polyhedron* which isolates the combinatorial information embodied in a convex polyhedron.

Definition 2.2. The underlying *abstract polyhedron* of a convex polyhedron P is the triple $(\mathcal{V}_P, \mathcal{F}_P, \mathcal{I}_P)$, where $\mathcal{V}_P$ is the set of vertices of P, $\mathcal{F}_P$ is the set of faces of P, and $\mathcal{I}_P \subset \mathcal{V}_P \times \mathcal{F}_P$ is the incidence relation between vertices and faces; that is, (v, f) is in $\mathcal{I}_P$ if and only if the vertex v lies on the face f.

Thus to say that two polyhedra P and Q have the same combinatorial structure is to say that their underlying abstract polyhedra are *isomorphic*; that is, there are bijections $\beta_V : \mathcal{V}_P \to \mathcal{V}_Q$ and $\beta_F : \mathcal{F}_P \to \mathcal{F}_Q$ that respect the incidence relation: (v, f) is in $\mathcal{I}_P$ if and only if $(\beta_V(v), \beta_F(f))$ is in $\mathcal{I}_Q$. Note that there is no need to record information about the edges; we leave it to the reader to verify that the edge data and incidence relations involving the edges can be recovered from the incidence structure $(\mathcal{V}_P, \mathcal{F}_P, \mathcal{I}_P)$. The cardinality of the set $\mathcal{I}_P$ is twice the number of edges of P, since

$$|\mathcal{I}_P| = \sum_{f \in \mathcal{F}_P} (\text{number of vertices on } f) = \sum_{f \in \mathcal{F}_P} (\text{number of edges on } f)$$

and the latter sum counts each edge of P exactly twice.

For the remainder of this section, we fix a convex polyhedron P and write V, E, and F for the numbers of vertices, edges, and faces of P, respectively. Let $\Pi = (\mathcal{V}, \mathcal{F}, \mathcal{I})$ be the underlying abstract polyhedron. We are interested in determining which sets of measurements are sufficient to determine P up to congruence. A natural place to start is with a naive dimension count: how many degrees of freedom does one have in specifying a polyhedron with the same combinatorial structure as P?

Definition 2.3. A *realization* of $\Pi = (\mathcal{V}, \mathcal{F}, \mathcal{I})$ is a pair of functions $(\alpha_\mathcal{V}, \alpha_\mathcal{F})$ where $\alpha_\mathcal{V} : \mathcal{V} \to \mathbf{R}^3$ gives a point for each v in $\mathcal{V}$, $\alpha_\mathcal{F} : \mathcal{F} \to \{\text{planes in } \mathbf{R}^3\}$ gives a plane for each f in $\mathcal{F}$, and the point $\alpha_\mathcal{V}(v)$ lies on the plane $\alpha_\mathcal{F}(f)$ whenever (v, f) is in $\mathcal{I}$.

Given any convex polyhedron Q together with an isomorphism $\beta : (\mathcal{V}_Q, \mathcal{F}_Q, \mathcal{I}_Q) \cong (\mathcal{V}, \mathcal{F}, \mathcal{I})$ of incidence structures we obtain a realization of Π, by mapping each vertex of P to the position of the corresponding (under β) vertex of Q and mapping each face of P to the plane containing the corresponding face of Q. In particular, P itself gives a realization of Π, and when convenient we'll also use the letter P for this realization. Conversely, while not every realization of Π comes from a convex polyhedron in this way, any realization of Π that's *sufficiently close* to P in the natural topology for the space of realizations gives—for example, by taking the convex hull of the image of α_V—a convex polyhedron whose underlying abstract polyhedron can be identified with Π. So the number of degrees of freedom is the dimension of the space of realizations of Π in a neighborhood of P.

Now we can count degrees of freedom. There are $3V$ degrees of freedom in specifying $\alpha_{\mathcal{V}}$ and $3F$ in specifying $\alpha_{\mathcal{F}}$. So if the $|\mathcal{I}| = 2E$ 'vertex-on-face' conditions are independent in a suitable sense then the space of all realizations of Π should have dimension $3V + 3F - 2E$ or—using Euler's formula—dimension $E + 6$. We must also take the congruence group into account: we have three degrees of freedom available for translations, and a further three for rotations. Thus if we form the quotient of the space of realizations by the action of the congruence group, we expect this quotient to have dimension E. This suggests that E measurements should suffice to pin down P up to congruence.

In the remainder of this section we show how to make the above naive dimension count rigorous, and how to identify specific sets of E measurements that suffice to determine congruence. The main ideas are: first, to use a combinatorial lemma (Lemma 2.7) to show that the linearizations of the vertex-on-face conditions are linearly independent at P, allowing us to use the inverse function theorem to show that the space of realizations really does have dimension $E + 6$ near P and to give an infinitesimal criterion for a set of measurements to be sufficient (Theorem 2.10), and second, to use an infinitesimal version of Cauchy's rigidity theorem to identify sufficient sets of measurements.

The various measurements that we're interested in can be thought of as real-valued functions on the space of realizations of Π (defined at least on a neighborhood of P) that are invariant under congruence. We single out one particular type of measurement: given two vertices v and w of P that lie on a common face, the *face distance* associated to v and w is the function that maps a realization $Q = (\alpha_{\mathcal{V}}, \alpha_{\mathcal{F}})$ of Π to the distance from $\alpha_{\mathcal{V}}(v)$ to $\alpha_{\mathcal{V}}(w)$. In other words, it corresponds to the measurement of the distance between the vertices of Q corresponding to v and w. The main result of this section is the following theorem.

Theorem 2.4. *Let P be a convex polyhedron with underlying abstract polyhedron $(\mathcal{V}, \mathcal{F}, \mathcal{I})$. Then there is a set S of face distances of P such that (i) S has cardinality E, and (ii) locally near P, the set S completely determines P up to congruence in the following sense: there is a positive real number ε such that for any convex polyhedron Q and isomorphism $\beta\colon (\mathcal{V}, \mathcal{F}, \mathcal{I}) \cong (\mathcal{V}_Q, \mathcal{F}_Q, \mathcal{I}_Q)$ of underlying abstract polyhedra, if*

1. *each vertex v of P is within distance ε of the corresponding vertex $\beta_{\mathcal{V}}(v)$ of Q, and*
2. *$m(Q) = m(P)$ for each measurement m in S,*

then Q is congruent to P.

Rephrasing:

Corollary 2.5. *Let P be a convex polyhedron with E edges. Then there is a set of E measurements that is sufficient to determine P up to congruence amongst all nearby convex polyhedra with the same combinatorial structure as P.*

We'll prove this theorem as a corollary of Theorem 2.10 below, which gives conditions for a set of measurements to be sufficient. We first fix some notation. Choose numberings $v_1, \dots, v_V$ and $f_1, \dots, f_F$ of the vertices and faces of Π, and write $(x_i(P), y_i(P), z_i(P))$ for the coordinates of vertex v_i of P. We translate P if necessary to ensure that no plane that contains a face of P passes through the origin. This allows us to give an equation for the plane containing f_j in the form $a_j(P)x +$

$b_j(P)y + c_j(P)z = 1$ for some nonzero triple of real numbers $(a_j(P), b_j(P), c_j(P))$; similarly, for any realization Q of Π that's close enough to P the ith vertex of Q is a triple $(x_i(Q), y_i(Q), z_i(Q))$ and the jth plane of Q can be described by an equation $a_j(Q)x + b_j(Q)y + c_j(Q)z = 1$. Hence the coordinate functions $(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, a_1, b_1, c_1, \ldots)$ give an embedding into $\mathbf{R}^{3V+3F}$ of some neighborhood of P in the space of realizations of Π.

For every pair (v_i, f_j) in $\mathcal{I}$ a realization Q should satisfy the 'vertex-on-face' condition

$$a_j(Q)x_i(Q) + b_j(Q)y_i(Q) + c_j(Q)z_i(Q) = 1.$$

Let $\phi_{i,j}$ be the function from $\mathbf{R}^{3V+3F}$ to $\mathbf{R}$ defined by

$$\phi_{i,j}(x_1, y_1, z_1, \ldots, a_1, b_1, c_1, \ldots) = a_j x_i + b_j y_i + c_j z_i - 1,$$

and let $\phi\colon \mathbf{R}^{3V+3F} \to \mathbf{R}^{2E}$ be the vector-valued function whose components are the $\phi_{i,j}$ as (v_i, f_j) runs over all elements of $\mathcal{I}$ (in some fixed order). Then a vector in $\mathbf{R}^{3V+3F}$ gives a realization of Π if and only if it maps to the zero vector under ϕ.

We next present a combinatorial lemma, Lemma 2.7, that appears as an essential component of many proofs of Steinitz's theorem characterizing edge graphs of polyhedra. (See Lemma 2.3 of [**9**], for example.) We give what we believe to be a new proof of this lemma. First, we make an observation that is an easy consequence of Euler's theorem.

Lemma 2.6. *Suppose that Γ is a planar bipartite graph of order r. Then there is an ordering $n_1, n_2, \ldots, n_r$ of the nodes of Γ such that each node n_i is adjacent to at most three preceding nodes.*

Proof. We give a proof by induction on r. If $r \leq 3$ then any ordering will do. If $r > 3$ then we can apply a standard consequence of Euler's formula (see, for example, Theorem 16 of [**3**]), which states that the number of edges in a bipartite planar graph of order $r \geq 3$ is at most $2r - 4$. If every node of Γ had degree at least 4 then the total number of edges would be at least $2r$, contradicting this result. Hence every nonempty planar bipartite graph has a node of degree at most 3; call this node n_r. Now remove this node (and all incident edges), leaving again a bipartite planar graph. By the induction hypothesis, there is an ordering $n_1, \ldots, n_{r-1}$ satisfying the conditions of the theorem, and then $n_1, \ldots, n_r$ gives the required ordering. ∎

Lemma 2.7. *Let P be a convex polyhedron. Consider the set $\mathcal{V} \cup \mathcal{F}$ consisting of all vertices and all faces of P. It is possible to order the elements of this set such that every vertex or face in this set is incident with at most three earlier elements of $\mathcal{V} \cup \mathcal{F}$.*

Proof. We construct a graph Γ of order $V + F$ as follows. Γ has one node for each vertex of P and one node for each face of P. Whenever a vertex v of P lies on a face f of P we introduce an edge of Γ connecting the nodes corresponding to v and f. Since P is convex, the graph Γ is planar; indeed, by choosing a point on each face of P, one can draw the graph Γ directly on the surface of P and then project onto the plane. (The graph Γ is known as the *Levi graph* of the incidence structure $\Pi = (\mathcal{V}, \mathcal{F}, \mathcal{I})$.) Now apply the preceding lemma to this graph. ∎

We now show that the functions $\phi_{i,j}$ are independent in a neighborhood of P. Write $D\phi(P)$ for the derivative of ϕ at P; as usual, we regard $D\phi(P)$ as a $2E$-by-$(3V + 3F)$ matrix with real entries.

Lemma 2.8. *The derivative $D\phi(P)$ has rank $2E$.*

In more abstract terms, this lemma implies that the space of all realizations of Π is, in a neighborhood of P, a smooth manifold of dimension $3V + 3F - 2E = E + 6$.

Proof. We prove that there are no nontrivial linear relations on the $2E$ rows of $D\phi(P)$. To illustrate the argument, suppose that the vertex v_1 lies on the first three faces and no others. Placing the rows corresponding to $\phi_{1,1}$, $\phi_{1,2}$, and $\phi_{1,3}$ first, and writing simply x_1 for $x_1(P)$ and similarly for the other coordinates, the matrix $D\phi(P)$ has the following structure:

$$D\phi(P) = \left(\begin{array}{ccc|c||ccccccccc|c}
a_1 & b_1 & c_1 & 0\ldots0 & x_1 & y_1 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots0 \\
a_2 & b_2 & c_2 & 0\ldots0 & 0 & 0 & 0 & x_2 & y_2 & z_2 & 0 & 0 & 0 & 0\ldots0 \\
a_3 & b_3 & c_3 & 0\ldots0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & z_3 & 0\ldots0 \\
\hline
0 & 0 & 0 & & \multicolumn{9}{c|}{} & \\
\vdots & \vdots & \vdots & * & \multicolumn{9}{c|}{*} & * \\
0 & 0 & 0 & & \multicolumn{9}{c|}{} &
\end{array}\right)$$

Here the vertical double bar separates the derivatives for the vertex coordinates from those for the face coordinates. Since the faces f_1, f_2, and f_3 cannot contain a common line, the 3-by-3 submatrix in the top left corner is nonsingular. Since v_1 lies on no other faces, all other entries in the first three columns are zero. Thus any nontrivial linear relation of the rows cannot involve the first three rows. So $D\phi(P)$ has full rank (that is, rank equal to the number of rows) if and only if the matrix obtained from $D\phi(P)$ by deleting the first three rows has full rank—that is, rank $2E - 3$. Extrapolating from the above, given any vertex that lies on exactly three faces, the three rows corresponding to that vertex may be removed from the matrix $D\phi(P)$, and the new matrix has full rank if and only if $D\phi(P)$ does. The dual statement is also true: exchanging the roles of vertex and face and using the fact that no three vertices of P are collinear we see that for any triangular face f we may remove the three rows corresponding to f from $D\phi(P)$, and again the resulting matrix has full rank if and only if $D\phi(P)$ does. Applying this idea inductively, if *every* vertex of P lies on exactly three faces (as in for example the regular tetrahedron, cube, or dodecahedron), or dually if *every* face of P is triangular (as in for example the tetrahedron, octahedron, or icosahedron) then the lemma is proved. For the general case, we choose an ordering of the faces and vertices as in Lemma 2.7. Then, starting from the end of this ordering, we remove faces and vertices from the list one-by-one, removing corresponding rows of $D\phi(P)$ at the same time. At each removal, the new matrix has full rank if and only if the old one does. But after removing all faces and vertices we're left with a 0-by-$2E$ matrix, which certainly has rank 0. So $D\phi(P)$ has rank $2E$. ∎

We now prove a general criterion for a set of measurements to be sufficient. Given the previous lemma, this criterion is essentially a direct consequence of the inverse function theorem.

Definition 2.9. A *measurement* for P is a smooth function m defined on an open neighborhood of P in the space of realizations of Π, such that m is invariant under rotations and translations.

Given any such measurement m, it follows from Lemma 2.8 that we can extend m to a smooth function on a neighborhood of P in $\mathbf{R}^{3V+3F}$. Then the derivative $Dm(P)$

is a row vector of length $3V + 3F$, well-defined up to a linear combination of the rows in $D\phi(P)$.

Theorem 2.10. *Let S be a finite set of measurements for Π near P. Let $\psi : \mathbf{R}^{3V+3F} \to \mathbf{R}^{|S|}$ be the vector-valued function obtained by combining the measurements in S, and write $D\psi(P)$ for its derivative at P, an $|S|$-by-$(3V + 3F)$ matrix whose rows are the derivatives $Dm(P)$ for m in S. Then the matrix*

$$D(\phi, \psi)(P) = \begin{pmatrix} D\phi(P) \\ D\psi(P) \end{pmatrix}$$

has rank at most $3E$, and if it has rank exactly $3E$ then the measurements in S are sufficient to determine congruence: that is, for any realization Q of Π, sufficiently close to P, if $m(Q) = m(P)$ for all m in S then Q is congruent to P.

Proof. Let $Q(t)$ be any smooth one-dimensional family of realizations of Π such that $Q(0) = P$ and $Q(t)$ is congruent to P for all t. Since each $Q(t)$ is a valid realization, $\phi(Q(t)) = 0$ for all t. Differentiating and applying the chain rule at $t = 0$ gives the matrix equation $D\phi(P)Q'(0) = 0$ where $Q'(0)$ is thought of as a column vector of length $3V + 3F$. The same argument applies to the map ψ: since $Q(t)$ is congruent to P for all t, $\psi(Q(t)) = \psi(P)$ is constant and $D\psi(P)Q'(0) = 0$. We apply this argument first to the three families where $Q(t)$ is P translated t units along the x-axis, y-axis, or z-axis respectively, and second when $Q(t)$ is P rotated by t radians around the x-axis, y-axis, or z-axis. This gives six column vectors that are annihilated by both $D\phi(P)$ and $D\psi(P)$. Writing G for the $(3V + 3F)$-by-6 matrix obtained from these column vectors, we have the matrix equation

$$\begin{pmatrix} D\phi(P) \\ D\psi(P) \end{pmatrix} G = 0.$$

It's straightforward to compute G directly; we leave it to the reader to check that the transpose of G is

$$\left(\begin{array}{ccccccc||ccccccc} 1 & 0 & 0 & 1 & 0 & 0 & \dots & -a_1^2 & -a_1b_1 & -a_1c_1 & -a_2^2 & -a_2b_2 & -a_2c_2 & \dots \\ 0 & 1 & 0 & 0 & 1 & 0 & \dots & -a_1b_1 & -b_1^2 & -b_1c_1 & -a_2b_2 & -b_2^2 & -b_2c_2 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots & -a_1c_1 & -b_1c_1 & -c_1^2 & -a_2c_2 & -b_2c_2 & -c_2^2 & \dots \\ 0 & -z_1 & y_1 & 0 & -z_2 & y_2 & \dots & 0 & -c_1 & b_1 & 0 & -c_2 & b_2 & \dots \\ z_1 & 0 & -x_1 & z_2 & 0 & -x_2 & \dots & c_1 & 0 & -a_1 & c_2 & 0 & -a_2 & \dots \\ -y_1 & x_1 & 0 & -y_2 & x_2 & 0 & \dots & -b_1 & a_1 & 0 & -b_2 & a_2 & 0 & \dots \end{array}\right)$$

For the final part of this argument, we introduce a notion of normalization on the space of realizations of Π. We'll say that a realization Q of Π is *normalized* if $v_1(Q) = v_1(P)$, the vector from $v_1(Q)$ to $v_2(Q)$ is in the direction of the positive x-axis, and $v_1(Q)$, $v_2(Q)$, and $v_3(Q)$ all lie in a plane parallel to the xy-plane, with $v_3(Q)$ lying in the positive y-direction from $v_1(P)$ and $v_2(P)$. In terms of coordinates we require that $x_1(P) = x_1(Q) < x_2(Q)$, $y_1(P) = y_1(Q) = y_2(Q) < y_3(Q)$, and $z_1(P) = z_1(Q) = z_2(Q) = z_3(Q)$. Clearly every realization Q of Π is congruent to a unique normalized realization, which we'll refer to as the *normalization* of Q. Note that the normalization operation is a continuous map on a neighborhood of P. Without loss of generality, rotating P around the origin if necessary, we may assume that P itself is normalized. The condition that a realization Q be normalized gives six more conditions on the coordinates of Q, corresponding to six extra functions $\chi_1, \ldots, \chi_6$, which we use to augment the function $(\phi, \psi)\colon \mathbf{R}^{3V+3F} \to \mathbf{R}^{2E+|S|}$ to a function $(\phi, \psi, \chi)\colon \mathbf{R}^{3V+3F} \to \mathbf{R}^{2E+|S|+6}$. These six functions are simply $\chi_1(Q) = x_1(Q) - x_1(P)$, $\chi_2(Q) = y_1(Q) - y_1(P)$, $\chi_3(Q) = z_1(Q) - z_1(P)$, $\chi_4(Q) = y_2(Q) - y_1(P)$, $\chi_5(Q) = z_2(Q) - z_1(P)$ and $\chi_6(Q) = z_3(Q) - z_1(P)$.

Claim 2.11. The 6-by-6 matrix $D\chi(P)G$ is invertible.

Proof. The matrix for G was given earlier; the product $D\chi(P)G$ is easily verified to be

$$\begin{pmatrix} 1 & 0 & 0 & 0 & z_1 & -y_1 \\ 0 & 1 & 0 & -z_1 & 0 & x_1 \\ 0 & 0 & 1 & y_1 & -x_1 & 0 \\ 0 & 1 & 0 & -z_2 & 0 & x_2 \\ 0 & 0 & 1 & y_2 & -x_2 & 0 \\ 0 & 0 & 1 & y_3 & -x_3 & 0 \end{pmatrix}$$

which has nonzero determinant $(y_3 - y_1)(x_2 - x_1)^2$. ∎

As a corollary, the columns of G are linearly independent, which proves that the matrix in the statement of the theorem has rank at most $3E$. Similarly, the rows of $D\chi(P)$ must be linearly independent, and moreover no nontrivial linear combination of those rows is a linear combination of the rows of $D(\phi, \psi)(P)$. Hence if $D(\phi, \psi)(P)$ has rank exactly $3E$ then the augmented matrix

$$\begin{pmatrix} D\phi(P) \\ D\psi(P) \\ D\chi(P) \end{pmatrix}$$

has rank $3E + 6$. Hence the map (ϕ, ψ, χ) has injective derivative at P, and so by the inverse function theorem the map (ϕ, ψ, χ) itself is injective on a neighborhood of P in $\mathbf{R}^{3V+3F}$. Now suppose that Q is a polyhedron as in the statement of the theorem. Let R be the normalization of Q. Then $\phi(R) = \phi(Q) = \phi(P) = 0$, $\psi(R) = \psi(Q) = \psi(P)$, and $\chi(R) = \chi(P)$. So if Q is sufficiently close to P, then by continuity of the normalization map R is close to P and hence $R = P$ by the inverse function theorem. So Q is congruent to $R = P$ as required. ∎

Definition 2.12. Call a set S of measurements *sufficient* for P if the conditions of the above theorem apply: that is, the matrix $D(\phi, \psi)(P)$ has rank $3E$.

Corollary 2.13. *Given a sufficient set S of measurements, there's a subset of S of size E that's also sufficient.*

Proof. Since the matrix $D(\phi, \psi)(P)$ has rank $3E$ by assumption, and the $2E$ rows coming from $D\phi(P)$ are linearly independent by Lemma 2.8, we can find E rows corresponding to measurements in S such that $D\phi(P)$ together with those E rows has rank $3E$. ∎

The final ingredient that we need for the proof of Theorem 2.4 is the infinitesimal version of Cauchy's rigidity theorem, originally due to Dehn, and later given a simpler proof by Alexandrov. We phrase it in terms of the notation and definitions above.

Theorem 2.14. *Let P be a convex polyhedron, and suppose that $Q(t)$ is a continuous family of polyhedra specializing to P at $t = 0$. Suppose that the real-valued function $m \circ Q$ is stationary (that is, its derivative vanishes) at $t = 0$ for each face distance m. Then $Q'(0)$ is in the span of the columns of G above.*

Proof. See Chapter 10, Section 1 of [**1**]. See also Section 5 of [**7**] for a short self-contained version of Alexandrov's proof. ■

Corollary 2.15. *The set of all face distances is sufficient.*

Proof. Let S be the collection of all face distances. Then Theorem 2.14 implies that for any column vector v such that $D(\phi, \psi)(P)v = 0$, v is in the span of the columns of G. Hence the kernel of the map $D(\phi, \psi)(P)$ has dimension exactly 6 and so by the rank-nullity theorem together with Euler's formula the rank of $D(\phi, \psi)(P)$ is $3V + 3F - 6 = 3E$. ■

Now Theorem 2.4 follows from Corollary 2.15 together with Corollary 2.13.

Finding an explicit sufficient set of E face distances is now a simple matter of turning the proof of 2.13 into a constructive algorithm. First compute the matrices $D\phi(P)$ and $D\psi(P)$ (the latter corresponding to the set S of *all* face distances). Initialize a variable T to the empty set. Now iterate through the rows of $D\psi(P)$: for each row, if that row is a linear combination of the rows of $D\phi(P)$ and the rows of $D\psi(P)$ corresponding to measurements already in T, discard it. Otherwise, add the corresponding measurement to T. Eventually, T will be a sufficient set of E face distances.

We have written a program in the Python programming language to implement this algorithm. This program is attached as Appendix B of the online version of this paper, available at www.arXiv.org. We hope that the comments within the program are sufficient explanation for those who wish to try it, and we thank Eric Korman for a number of these comments.

The algorithm works more generally. Given any sufficient set of measurements (in the sense of Definition 2.12), which may include angles, it will extract a subset of E measurements which is sufficient. It will also find a set of angle measurements which determines a polyhedron up to similarity. As an example we consider a similarity calculation for a dodecahedron. Our allowed measurements will be the set of angles formed by pairs of lines from a vertex to two other vertices on a face containing that vertex. In Figure 1 we show a set of 29 such angles which the program determined to characterize a dodecahedron up to similarity.

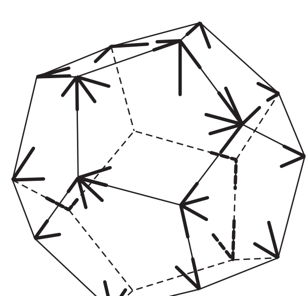

Figure 1. A set of 29 angles which locally determine a dodecahedron up to similarity.

There is no restriction of the method to Platonic solids. Data for a number of other examples can be found in the program listing. Among these examples are several which are nonconvex. The program appears to give a result for many of these, but we have not extended the theory beyond the convex case.

2.1. Related Work. The results and ideas of Section 2 are, for the most part, not new. The idea of an abstract polyhedron represented by an incidence structure, and its realizations in $\mathbf{R}^3$, appears in Section 2 of [**14**]. In Corollary 15 of that paper, Whiteley proves that the space of realizations of a 'spherical' incidence structure (equivalent to an incidence structure arising from a convex polyhedron) has dimension E. The essential combinatorial content of the proof of Lemma 2.8 is often referred to as 'Steinitz's lemma,' and a variety of proofs appear in the literature ([**10**], [**8**]); we believe that the proof above is new.

3. HOW MANY MEASUREMENTS ARE ENOUGH? Theorem 2.4 provides an upper bound for the number of distance/angle measurements needed to describe a polyhedron with given combinatorial structure. But it turns out that many interesting polyhedra can be described with fewer measurements. In particular, a cube can be determined by 9 distance or angle measurements instead of 12. Also, 10 distance measurements (no angles used) suffice.

This phenomenon appears already in dimension two. Much of this section deals with polygons and sets of points in the plane. In this simpler setting one can often find precisely the smallest number of measurements needed. Extending these results to polyhedra, with or without fixing their combinatorial structure, certainly deserves further study.

So, we turn our attention to polygons. Unlike polyhedra, their combinatorial structure is relatively simple, especially if we restrict to the case of convex polygons. One can argue about the kinds of measurements that should be allowed, but the following three seem the most natural to us.

Definition 3.1. Suppose that $A_1 A_2 \cdots A_n$ is a convex polygon. (More generally, suppose $\{A_1, A_2, \ldots, A_n\}$ is a sequence of distinct points on the plane). Then the following are called simple measurements:

1. distances $\left|A_i A_j\right|$, $i \neq j$,
2. angles $\angle A_i A_j A_k$ for i, j, k distinct,
3. 'diagonal angles' between $A_i A_j$ and $A_k A_l$.

Other quantities might also be considered, like the distance from A_i to $A_j A_k$, but in practice these would require several measurements.

It is natural to ask how many simple measurements are needed to determine a polygon up to isometry. In the case of a triangle the answer is 3. As mentioned earlier, there are a few ways to do it. We note that two sides of a triangle and an angle adjacent to one of the sides do not always determine the triangle uniquely, but they do determine it locally (as in part (ii) of Theorem 2.4).

It is easy to see that for any n-gon $P = A_1 \cdots A_n$ the following $(2n-3)$ measurements suffice: the $n-1$ distances $|A_1A_2|, |A_2A_3|, \ldots, |A_{n-1}A_n|$, together with the $n-2$ angles $\angle A_1A_2A_3, \ldots, \angle A_{n-2}A_{n-1}A_n$. Observe also that instead of using distances and angles, one can use only distances, by substituting $|A_{i-1}A_{i+1}|$ for $\angle A_{i-1}A_iA_{i+1}$.

The following theorem implies that for most polygons one cannot get away with fewer measurements. To make this rigorous, we will assume that $A_1 = (0, 0)$, and that $A_2 = (x_2, 0)$ for some $x_2 > 0$. It then becomes obvious that to each polygon we can associate a point in $\mathbf{R}^{2n-3}$, corresponding to the undetermined coordinates $x_2, x_3, y_3, \ldots, x_n, y_n$ with $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$.

Theorem 3.2. *Denote by S the set of all points in $\mathbf{R}^{2n-3}$ obtained by the procedure above from those polygons that can be determined up to isometry by fewer than $2n-3$ measurements. Then S has measure zero.*

Proof. Observe that any set of $2n-3$ specific measurements is a smooth map from $\mathbf{R}^{2n-3}$ to itself. There are only a finite number of such maps, with measurements chosen from the types 1, 2, and 3. At a noncritical point, the map has an inverse. The result is then a consequence of Sard's theorem (Chapter 2, Theorem 8 of [**12**]). ■

The above theorem is clearly not new, and we claim no originality for its statement or proof. For $n = 3$ it implies that for a generic triangle one needs at least three simple measurements. This is true for any triangle, because two measurements are clearly not enough. For $n = 4$ the theorem implies that a generic convex quadriateral requires five measurements. One can be tempted to believe that no quadrilateral can be described by fewer than five measurements. In fact, the first reaction of most mathematicians seems to be that if a general case requires five measurements, all special cases do so as well. The following example seems to confirm this intuition.

Example 3.3. Suppose $P = A_1A_2A_3A_4$ is a quadrilateral. Suppose $\angle A_2A_1A_4 = \alpha_1$, $\angle A_3A_4A_1 = \alpha_2$, $|A_1A_2| = d_1$, $|A_3A_4| = d_2$, and $\angle A_1A_2A_3 = \alpha_3$. For most polygons P these five measurements are sufficient to determine P up to congruence. However for some P they are not sufficient. For example, if P is a square, there are infinitely many polygons with the same measurements: all rectangles with the same $|A_1A_2|$.

The above example suggests that the special polygons require at least as many measurements as the generic ones. So $(2n-3)$ should be the smallest number of simple measurements required for any n-gon. This reasoning is reinforced by the argument in Section 1 that a square is determined by five unknowns, and so we need five measurements.

The simplest way to raise doubt about this is to observe that the single equation $x^2 + y^2 = 0$ determines both x and y. (Algebraic geometers should note that we are dealing here with real, not complex, varieties.) In fact, the intuition that $2n-3$ is the minimum number of required measurements in all cases is very far from the truth: many interesting n-gons can be determined by fewer than $(2n-3)$ simple measurements.

The simplest example in this direction is not usually considered a polygon. Suppose $A_1A_2A_3A_4$ is a set of four points in the plane that lie on the same line, in the natural order. Then the four distances $|A_1A_4|$, $|A_1A_2|$, $|A_2A_3|$, and $|A_3A_4|$ determine the configuration up to isometry. This means that any subset of four points in the plane with the same corresponding measurements is congruent to $A_1A_2A_3A_4$. The proof of this is, of course, a direct application of the triangle inequality:

$$|A_1A_4| \leq |A_1A_3| + |A_3A_4| \leq |A_1A_2| + |A_2A_3| + |A_3A_4|.$$

One can think of A_1A_2, A_2A_3, A_3A_4, and A_1A_4 as rods of fixed length. When $|A_1A_4| = |A_1A_2| + |A_2A_3| + |A_3A_4|$, the configuration allows no room for wiggling; all rods must be lined up. This degenerate example easily generalizes to n points and n distance measurements. More surprisingly, it generalizes to some convex polygons.

Definition 3.4. A polygon is called *exceptional* if it can be described locally by fewer than $2n - 3$ measurements.

Obviously, no triangles are exceptional, in the above sense. But already for quadrilaterals, there exist some exceptional ones that could be defined by four rather than five measurements. The biggest surprise for us was the following observation.

Proposition 3.5. *All squares are exceptional! Specifically, for a square ABCD the following four measurements distinguish it among all quadrilaterals:*

$$|AB|, |AC|, |AD|, \angle BCD.$$

Proof. Suppose $A'B'C'D'$ is another quadrilateral with $|A'B'| = |AB|$, $|A'C'| = |AC|$, $|A'D'| = |AD|$, and $\angle B'C'D' = \angle BCD$. If $|AB| = d$, this means that $|A'B'| = |A'D'| = d$, $|A'C'| = d\sqrt{2}$, and $\angle B'C'D' = \frac{\pi}{2}$. We claim that the quadrilateral $A'B'C'D'$ is congruent to the square $ABCD$. Indeed, consider A' fixed. Then B' and D' lie on the circle of radius d centered at A', while C' lies on the circle of radius $d\sqrt{2}$ centered at A'. Then $\frac{\pi}{2}$ is the maximal possible value of the angle $B'C'D'$ and it is achieved when $\angle A'B'C' = \angle A'D'C' = \frac{\pi}{2}$, making $A'B'C'D'$ a square, with the side $|A'B'| = d$. ■

The idea of showing that a particular polygon occurs when some angle or length is maximized within given constraints, and that there is only one such maximum, is at the heart of all of the examples in this section. It is related to the notion of second-order rigidity of tensegrity networks; see [**6**] and [**11**].

One immediate generalization of this construction is the following 3-parameter family of exceptional quadrilaterals.

Proposition 3.6. *Suppose that ABCD is a convex quadrilateral with $\angle ABC = \angle CDA = \frac{\pi}{2}$. Then ABCD is exceptional. It is determined up to congruence by the following four measurements: $|AB|$, $|AD|$, $|AC|$, and $\angle BCD$.*

Proof. Consider an arbitrary quadrilateral $ABCD$ with the same four measurements as the quadrilateral that we are aiming for (note that we are not assuming that $\angle ABC = \angle CDA = \frac{\pi}{2}$). We can consider the points A and C fixed in the plane. Then the points B and D lie on two fixed circles around A, with radius $|AB|$ and radius $|AD|$. Among all such pairs of points on these circles, on opposite sides of AC, the maximum possible value of $\angle BCD$ is obtained for only one choice of B and D. This is the choice which makes CB and CD tangent to the circles at B and D, and so $\angle ABC$ and $\angle ADC$ are right angles, and the quadrilateral $ABCD$ is congruent to the one we are aiming for. ■

This family of quadrilaterals includes all rectangles. In a sense it is the biggest possible: one cannot hope for a four-parameter family requiring just four measurements. Another such family is given below. Note that it includes all rhombi that are not squares.

Proposition 3.7. *For given points B and D, and given acute angles θ_1 and θ_2, choose A and C so that $\angle DAB = \theta_1$, $\angle DCB = \theta_2$, and $|AC|$ is as large as possible. Then this determines a unique quadrilateral ABCD, up to congruence, and this quadrilateral has the property that $|AB| = |AD|$ and $|CB| = |CD|$.*

We leave the proof to the reader. It implies that for the set of quadrilaterals $ABCD$ such that AC is a perpendicular bisector of BD, and the angles $\angle DAB$ and $\angle DCB$ are acute, the measurements $|BD|$, $|AC|$, $\angle DAB$, and $\angle DCB$ are sufficient to determine $ABCD$.

As David Allwright pointed out to the authors, one can further extend this example to the situation when $\angle DAB + \angle DCB < \pi$.

The existence of so many exceptional quadrilaterals suggests that for bigger n it may be possible to go well below the $(2n-3)$ measurements. This is indeed correct: for every n there are polygons that can be defined by just n measurements.

Proposition 3.8. *Suppose that $A_1A_2\cdots A_n$ is a convex polygon, with*

$$\angle A_1A_2A_3 = \angle A_1A_3A_4 = \cdots = \angle A_1A_kA_{k+1} = \cdots = \angle A_1A_{n-1}A_n = \frac{\pi}{2}.$$

Then $A_1A_2\cdots A_n$ is exceptional; moreover the distances $|A_1A_2|$, $|A_1A_n|$ and the angles $\angle A_kA_{k+1}A_1$, $2 \le k \le n-1$ determine the polygon. Note that many such polygons exist for every n.

Proof. Suppose $\angle A_kA_{k+1}A_1 = \alpha_k$, $2 \le k \le (n-1)$. Then because of the law of sines for the triangles $\triangle A_1A_kA_{k+1}$, we obtain the following sequence of inequalities:

$$|A_1A_n| \le \frac{|A_1A_{n-1}|}{\sin\alpha_{n-1}} \le \frac{|A_1A_{n-2}|}{\sin\alpha_{n-1}\cdot\sin\alpha_{n-2}} \le \cdots \le \frac{|A_1A_2|}{\sin\alpha_{n-1}\cdot\cdots\cdot\sin\alpha_2}.$$

Equality is achieved if and only if all angles $A_1A_kA_{k+1}$ are right angles, which implies the result. ■

One can ask whether an even smaller number of measurements might work for some very special polygons. The following theorem shows that in a very strong sense the answer is negative.

Theorem 3.9. *For any sequence of distinct points $A_1, A_2, \ldots, A_n$ in the plane, with $n \ge 3$, one needs at least n simple measurements to determine it up to plane isometry.*

Proof. The result is obvious if $n = 3$, so assume that $n > 3$. At least one distance measurement is needed, and we can assume that it is $|A_1A_2|$. We assume that A_1A_2 is fixed, so the positions of the other $n-2$ points determine the set up to isometry. We identify the ordered set of coordinates of these points with a point in $\mathbf{R}^{2(n-2)} = \mathbf{R}^{2n-4}$. The set of all sequences of distinct points then corresponds to an open set $U \subset \mathbf{R}^{2n-4}$. Each measurement of type 1, 2, or 3 determines a smooth submanifold $V \subset U$. The following observation is the main idea of the proof.

Lemma 3.10. *Suppose that $x \in V$. Then there exists an affine subspace W of $\mathbf{R}^{2n-4}$, of dimension at least $2n-6$, which contains x and is such that for some open ball B containing x we have $W \cap B \subset V$.*

Proof. The proof of this lemma involves several different cases, depending on the kind of measurement used and whether A_1 and/or A_2 are involved. We give some examples, the other cases being similar. First suppose that the polygon is the unit square, with vertices at $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (1,1)$, and $A_4 = (0,1)$. Suppose that the measurement is the angle $\angle A_2A_1A_3$. With A_1 and A_2 fixed, we are free to move

A_3 along the line $y = x$, and A_4 arbitrarily. In this case, then, W could be three dimensional, one more than promised by the lemma. Second, again with the unit square, suppose that the measurement is the distance from A_3 to A_4. In this case, we can move A_3 and A_4 the same distance along parallel lines. Thus,

$$W = \{(1 + c, 1 + d, c, 1 + d)\}$$

is the desired two-dimensional affine space. Finally, suppose that $n \geq 5$ and that the measurement is the angle between $A_1 A_k$ and $A_l A_m$ where $1, 2, k, l, m$ are distinct. In this case we can move A_l arbitrarily, giving two free parameters, and we can move A_m so that $A_l A_m$ remains parallel to the original line containing these points. Since the length $|A_l A_m|$ can be changed, this gives a third parameter. Then the length $A_1 A_k$ can be changed, giving a fourth, and the remaining $n - 5$ points can be moved, giving $2n - 10$ more dimensions. In this case, W is $2n - 6$ dimensional. We leave other cases to the reader. ∎

Continuing the proof of Theorem 3.9, we first show that $k \leq n - 3$ additional measurements are insufficient to determine the points $A_1, \ldots, A_n$. Recall that one measurement, $|A_1 A_2|$, was already used. Suppose that a configuration $x \in \mathbf{R}^{2n-4}$ lies in $V = \cap_{i=1}^{k} V_i$, where V_i is the submanifold defined by the ith measurement. We will prove that x is not an isolated point in V.

Denote the affine subspaces obtained from Lemma 3.10 corresponding to V_i and x by W_i. Let $W = W_1 \cap \cdots \cap W_k$. Since each W_i has dimension at least $2n - 6$, its codimension in $\mathbf{R}^{2n-4}$ is less than or equal to 2.[3] Hence

$$\operatorname{codim} W \leq \operatorname{codim} W_1 + \cdots + \operatorname{codim} W_k \leq 2k \leq 2\,(n - 3)$$

so W must have dimension at least 2. Some neighborhood of x in W is contained in all of the submanifolds $V_1, \ldots, V_k$. This neighborhood is then contained in V, showing that x is not isolated.

The case $k = n - 2$ is trickier, because the dimensional count does not work in such a simple way. If $W = W_1 \cap \cdots \cap W_k \neq \{x\}$, then W has dimension at least 1, and we can argue as before that x is not isolated in V. Now suppose that $W = \{x\}$. Then since $W' = W_2 \cap \cdots \cap W_k$ has dimension at least 2, it must be a two-dimensional affine subspace intersecting the codimension-2 affine subspace W_1 at the point x.

We now consider this situation in more detail. Denote the measurement defining V_1 by $f_1 : \mathbf{R}^{2n-4} \to \mathbf{R}$. The linearization of f_1 at x is its derivative, $Df_1\,(x)$. For measurements of types 1, 2, or 3 above it is not hard to see that $Df_1\,(x) \neq \mathbf{0}$. If X is the null space of the $1 \times (2n - 4)$ matrix $Df_1\,(x)$, then the tangent space of V_1 at x can be defined as the affine subspace $T_x\,(V_1) = x + X$. This has dimension $2n - 5$. Since W_1 has codimension 2, it is a codimension-1 subspace of $T_x\,(V_1)$. W' is not contained in $T_x\,(V_1)$, because W' and W_1 span the entire $\mathbf{R}^{2n-4}$. Thus, the intersection $W' \cap T_x\,(V_1)$ is one-dimensional.

We wish to show that x is not isolated in $V_1 \cap W'$. We consider the map $g_1 = f_1|_{W'}$. Because W' is not contained in $T_x\,(V_1)$, $Dg_1(x) \neq 0$. So the implicit function theorem implies that x is not an isolated zero of g_1 in W', proving the theorem. ∎

Similar ideas can be used to construct other interesting examples of exceptional polygons and polyhedra. The following are worth mentioning.

[3]If V is an affine subspace of $\mathbf{R}^m$, then it is of the form $x + X$, where X is a subspace of $\mathbf{R}^m$. If the orthogonal complement of X in $\mathbf{R}^m$ is Y, then the codimension of V is the dimension of Y.

1. There exist tetrahedra determined by just five measurements, instead of the generic six. In particular, if measurement of dihedral angles is permitted, the regular tetrahedron can be determined using five measurements. It seems unlikely that four would ever work.
2. There exist 5-vertex convex polyhedra that are characterized by just five measurements. Moreover, four of the vertices are on the same plane, but, unlike in the beginning of the paper, we do not have to assume this a priori! To construct such an example, we start with an exceptional quadrilateral $ABCD$ as in Proposition 3.6, with $|AD| < |AB|$. We then add a vertex E outside of the plane of $ABCD$ so that $\angle ADE = \angle AEB = \frac{\pi}{2}$. There are obviously many such polyhedra. Now notice that the distances $|AD|$, $|AC|$ and angles $\angle AED$, $\angle ABE$, $\angle BCD$ completely determine the configuration.
3. As pointed out earlier, using four measurements for a square, one can determine the cube with just nine distance or angle measurements. Interestingly, one can also use 10 distance measurements for a cube. If $A_1B_1C_1D_1$ is its base and $A_2B_2C_2D_2$ is a parallel face, then the six distances between A_1, B_1, D_1, and A_2 completely fix the relative position of these vertices. Then $|A_1C_2|$, $|B_2C_2|$, $|D_2C_2|$, and $|C_1C_2|$ determine the cube, because

$$|A_1C_2|^2 \le |B_2C_2|^2 + |D_2C_2|^2 + |C_1C_2|^2,$$

with equality only when the segments B_2C_2, D_2C_2, and C_1C_2 are perpendicular to the corresponding faces of the tetrahedron $A_1B_1D_1A_2$.

The last example shows that the problem is not totally trivial even when we only use distance measurements. It is natural to ask for the smallest number of distance measurements needed to fix a nondegenerate n-gon in the plane. We have the following theorem.

Theorem 3.11. *Suppose $A_1, A_2, \ldots, A_n$ is a sequence of points in the plane, with no three of them on the same line. Then one needs at least* $\min(2n-3, \frac{3n}{2})$ *distance measurements to determine it up to a plane isometry.*

Proof. We use induction on n. For $n = 1$ and $n = 2$ the statement is trivial. Suppose that $n \ge 3$ and that the statement is true for all sets of fewer than n points, and that there is a nondegenerate configuration of n points, $A_1, A_2, \ldots, A_n$, that is fixed by k measurements where $k < 2n - 3$ and $k < \frac{3n}{2}$. Because $k < \frac{3n}{2}$, there is some point A_i which is being used in less than three measurements. If it is only used in one measurement, the configuration is obviously not fixed. So we can assume that it is used in two distance measurements, $|A_iA_1|$ and $|A_iA_2|$. Because A_1, A_2, and A_i are not on the same line, the circle with radius $|A_iA_1|$ around A_1 and the circle with radius $|A_iA_2|$ around A_2 intersect transversally at A_i. Now remove A_i and these two measurements. By the induction assumption, the remaining system of $(n-1)$ points admits a small perturbation. This small perturbation leads to a small perturbation of the original system. ∎

For $n \le 7$ the above bound coincides with the upper bound $(2n-3)$ for the minimal number of measurements (see note after Theorem 1). One can show that for $n \ge 8$, $\lceil \frac{3n}{2} \rceil$ is the optimal bound, with the regular octagon being the simplest 'distance-exceptional' polygon. In fact there are at least two different ways to define the regular octagon with 12 distance measurements.

Example 3.12. Suppose $A_1A_2A_3 \cdots A_8$ is a regular octagon. Its eight sides and the diagonals $|A_1A_5|$, $|A_8A_6|$, $|A_2A_4|$, and, finally, $|A_3A_7|$ determine it among all octagons. A proof is elementary but too messy to include here; its main idea is that $|A_3A_7|$ is the biggest possible with all other distances fixed. In fact, one can show that the distance between the midpoints of A_1A_5 and A_6A_8 is maximal when $\angle A_5A_1A_8 = \angle A_6A_5A_1 = \frac{3\pi}{8}$.

Example 3.13. Suppose $A_1A_2A_3 \cdots A_8$ is a regular octagon. Its 8 sides and the four long diagonals determine it among all octagons. This follows from [**5**], Corollary 1 to Theorem 5. In fact, this example can be generalized to any regular $(2n)$-gon: a tensegrity structure with cables for the edges and struts for the long diagonals has a proper stress due to its symmetry and thus is rigid. This provides an optimal bound for the case of an even number of points, and one can get the result for an odd number of points just by adding one point at a fixed distance to two vertices of the regular $(2n)$-gon. We should note that tensegrity networks and their generalizations have been extensively studied; see for example [**6**].

There are many open questions in this area, some of which could be relatively easy to answer. For example, we do not know whether either of the results for the cube (9 measurements, or 10 distance measurements) is best possible. Also, it is relatively easy to determine a regular hexagon by 7 measurements, but it is not known if 6 suffice.

One can also try to extend the general results of this section to the three-dimensional case. The methods of Theorem 3.11 produce, for any sufficiently large n, a lower estimate of $2n$ for the number of distance measurements needed to determine a set of n points, no four of which lie on the same plane. One should also note that many of the exceptional polygons that we have constructed, for instance the square, are unique even when considered in three-dimensional space: the measurements guarantee their planarity. The question of determining the smallest number of measurements for a polyhedron, using distances and angles, with planarity conditions for the faces, seems to be both the hardest and the most interesting. But even without the planarity conditions or without the angles the answer is not known.

ACKNOWLEDGMENTS. The authors wish to thank David Allwright for many valuable comments on a previous version of the manuscript. We also thank Joseph O'Rourke and the anonymous referee for their exceptionally thorough and detailed reviews and for pointing out a number of relevant references to the literature.

REFERENCES

1. A. D. Alexandrov, *Convex Polyhedra* (trans. from 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze, and A. B. Sossinsky), Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.
2. D. Allwright, Quadrilateral-faced hexahedrons with all face-diagonals equal (preprint).
3. B. Bollobás, *Modern Graph Theory*, vol. 184, Graduate Texts in Mathematics, Springer-Verlag, New York, 1998.
4. R. Connelly, A counterexample to the rigidity conjecture for polyhedra, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977) 333–338. `doi:10.1007/BF02684342`
5. ———, Rigidity and energy, *Invent. Math.* **66** (1982) 11–332. `doi:10.1007/BF01404753`
6. R. Connelly and W. Whiteley, Second-order rigidity and prestress stability for tensegrity frameworks, *SIAM J. Discrete Math.* **9** (1996) 453–491. `doi:10.1137/S0895480192229236`
7. H. Gluck, Almost all simply connected closed surfaces are rigid, in *Geometric Topology—Park City 1974*, Lecture Notes Math., vol. 438, Springer-Verlag, New York, 1975, 225–239.
8. B. Grünbaum, *Convex Polytopes*, vol. 221, Graduate Texts in Mathematics, Springer-Verlag, New York, 2003.
9. ———, Graphs of polyhedra; polyhedra as graphs, *Discrete Math.* **307** (2007) 445–463. `doi:10.1016/j.disc.2005.09.037`

10. L. A. Lyusternik, *Convex Figures and Polyhedra*, Dover, Mineola, NY, 1963.
11. B. Roth and W. Whiteley, Tensegrity frameworks, *Trans. Amer. Math. Soc.* **265** (1981) 419–446. `doi:10.2307/1999743`
12. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. I, 3rd ed., Publish or Perish, Houston, TX, 1999.
13. T. Tarnai and E. Makai, A movable pair of tetrahedra, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **423** (1989) 419–442. `doi:10.1098/rspa.1989.0063`
14. W. Whiteley, How to describe or design a polyhedron, *Journal of Intelligent and Robotic Systems* **11** (1994) 135–160. `doi:10.1007/BF01258299`

ALEXANDER BORISOV received an M.S. in mathematics from Moscow State University and a Ph.D. from Penn State University. He is currently an assistant professor at the University of Pittsburgh. He has authored and co-authored papers on a wide range of topics in algebraic geometry, number theory, and related areas of pure mathematics.
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
borisov@pitt.edu

MARK DICKINSON received his Ph.D. from Harvard University in 2000. He has held positions at the University of Michigan and the University of Pittsburgh, and is currently a researcher in computational homological algebra at the National University of Ireland, Galway. His research interests include algebraic number theory, representation theory, and formal mathematical proofs using a proof assistant. He is also a keen programmer, and is one of the core developers of the popular 'Python' programming language.
Mathematics Department, National University of Ireland, Galway, Ireland
dickinsm@gmail.com

STUART HASTINGS has worked almost entirely in the area of ordinary differential equations (which these days one has to mislabel as "dynamical systems" to sound respectable). A student of Norman Levinson at M.I.T., he had positions at Case Western Reserve and SUNY, Buffalo, before coming to Pittsburgh over twenty years ago. He is very happy to have his first publication in geometry, which he credits to (a) his grandchildren, whose existence prompted a foray into "mathematical woodworking," and (b) departmental teas, without which he might never have discussed the problem of measuring polyhedra with two talented young researchers whose knowledge and ability in geometry greatly exceeded his own.
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
sph@pitt.edu

Mathematics Is . . .

"Mathematics is that form of intelligence in which we bring the objects of the phenomenal world under the control of the conception of quantity."

G. H. Howison, The departments of mathematics, and their mutual relations, *Journal of Speculative Philosophy* **5** (1871) 164.

—Submitted by Carl C. Gaither, Killeen, TX