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SINGULAR TORIC FANO VARIETIES

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ABSTRACT. The authors prove that the number of types of toric Fano varieties with certain constraints on the singularities is finite.

INTRODUCTION

A *toric variety* of dimension n over an arbitrary algebraically closed field k is a normal variety X that is an equivariant compactification of the n-dimensional torus $T \cong (k^*)^n$, where k^* is the multiplicative group of the field k. Let $M \cong \mathbb{Z}^n$ be the character lattice of the torus T. Then every toric variety X is given by a fan Σ in the space $N \otimes \mathbb{Q}$, where N is the lattice dual to M (see [5]).

A *Fano variety* is a normal projective variety X such that some positive integer multiple of the anticanonical divisor $-K_X$ is an ample Cartier divisor. This definition imposes the $\mathbb{Q}$-Gorenstein property for the singularities of a Fano variety.

Let X be the toric variety given by a fan Σ in $N \otimes \mathbb{Q}$. Let A_i be the closest points to 0 of the one-dimensional cones of the fan Σ. Then X is a Fano variety if and only if the A_i are the vertices of a convex polyhedron P in the lattice N (see [5]).

Definition. A Fano variety X is called a *variety with log-terminal singularities* if for an arbitrary resolution of singularities $\pi\colon Y \to X$ all the a_i in the formula

$$K_Y = \pi^* K_X + \sum_i a_i E_i, \tag{1}$$

relating the classes of the canonical divisors K_X, K_Y and the exceptional divisors E_i, are bigger than -1.

One can show that any toric Fano variety is a variety with log-terminal singularities. In this paper we prove the following result.

Theorem. *For any n and $\varepsilon > 0$ there exist only a finite number of types of n-dimensional toric Fano varieties X such that for any resolution of singularities $\pi\colon Y \to X$ the inequality $a_i \geq -1 + \varepsilon$ holds for all i, where the a_i are defined in* (1).

In §1 this theorem is reformulated in the language of convex polyhedra in n-dimensional space. In §§2–4 it is proved in the special case of simplices, which corresponds to the case of varieties with Picard number 1. In §5, using the results proved earlier, we finish the proof in the general case.

From the proof of this theorem we can extract an algorithm that allows us, with the help of a computer, to classify toric Fano varieties. We have obtained a classification of three-dimensional toric Fano varieties with canonical and terminal singularities and Picard number 1.

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Three-dimensional toric Fano varieties with terminal singularities and Picard number 1 are the weighted projective spaces with weights

$$(1, 1, 1, 1), \quad (2, 1, 1, 1), \quad (3, 2, 1, 1), \quad (5, 3, 2, 1),$$
$$(5, 4, 3, 1), \quad (7, 5, 3, 2), \quad (7, 5, 4, 3)$$

and the toric variety that is given in N by the simplex with vertices

$$(1, 0, 1), \quad (-2, 1, 1), \quad (1, -2, 0), \quad (0, 1, -2).$$

There are 225 types of three-dimensional toric Fano varieties with canonical singularities and Picard number 1.

The problem of whether the number of types of toric Fano varieties with given constraints on the singularities is of interest because of the minimal model program. In 1982, V. V. Batyrev proved that there exists a constant $C(n, d)$ which bounds the degree of all n-dimensional toric Fano varieties with the condition that dK_X is a Cartier divisor (see [2]).

Similar results were obtained by V. A. Alekseev and V. V. Nikulin (see [1], [6], and [7]) for arbitrary (not necessarily toric) surfaces. There are reasons to believe that the theorem proved in this paper also has an analog for surfaces.

A classification of all three-dimensional toric Fano varieties in the smooth case was obtained by Batyrev [3] and by K. Watanabe and M. Watanabe [8]. See also [4].

The authors are grateful to V. V. Batyrev for posing the problem and for his help.

§1. A reformulation of the theorem

Without loss of generality we may assume that $\varepsilon < 1$.

Proposition 1. *Let X be a toric Fano variety given by a fan Σ in $N \otimes \mathbb{Q}$, and A_i the closest integral points to 0 of the one-dimensional cones of the fan Σ. Let P be the convex polyhedron with vertices A_i. If for any resolution of singularities $\pi\colon Y \to X$ all the a_i in* (1) *are bigger than $-1+\varepsilon$, then the polyhedron εP obtained from P by the homothety with center at zero and coefficient ε contains no integral points* (*an integral point in $N \otimes \mathbb{Q}$ is just a point of the lattice N*).

Proof. Consider the toric resolution of singularities $\pi\colon Y = Y_{\Sigma'} \to X = X_\Sigma$ (see [5]). Suppose that the one-dimensional cones l_i of the fan Σ' correspond to the exceptional divisors E_i. Let B_i denote the closest integral points of the rays l_i to 0.

Invariant divisors on a toric variety correspond to functions linear on each of the cones of the fan (see [5]).

Let K_X be an invariant canonical divisor on X such that $\varphi_{K_X}(A_i) = -1$. Then $K_Y = \pi^* K_X + \Sigma_i a_i E_i$ is an invariant canonical divisor on Y, and $\varphi_{K_Y}(A_i) = -1$. It is now easy to see that $\varphi_{K_Y}(B_i) = -1$ for all i. Notice that

$$\begin{aligned} -1 = \varphi_{K_Y}(B_i) &= \varphi_{K_X}(B_i) + \sum_j a_j \varphi_{E_j}(B_i) \\ &= \varphi_{K_X}(B_i) + a_i > \varphi_{K_X}(B_i) - 1 + \varepsilon. \end{aligned}$$

Therefore $\varphi_{K_X}(B_i) < -\varepsilon$. Since $\varphi_{K_X}(A_i) = -1$, we conclude that B_i is not contained in εP.

Suppose now that there is an integral point $A \neq 0$ inside εP. Since the variety Y is smooth, we have

$$\overline{OA} = \sum_j \alpha_j \overline{OA}_j + \sum_j \beta_j \overline{OB}_j, \qquad \alpha_j, \beta_j \in N, \ O = 0,$$

where A_j and B_j lie on the rays bounding the cone from Σ' that contains A.

This, however, is impossible because B_j and A_j are not contained in εP. The proposition is proved.

Thus to prove the theorem it will suffice to prove the following assertion: up to the action of $GL_n(\mathbb{Z})$, in $\mathbb{Z}^n$ there are only a finite number of convex polyhedra P containing 0 and such that εP contains no nonzero points of the lattice $\mathbb{Z}^n$. It is this assertion that will be proved in the remaining four sections.

§2. Simplices generating the lattice

We fix a lattice $\mathbb{Z}^n$ with zero element O. When speaking of points and simplices we shall always assume that the points and the vertices of the simplices under consideration are contained in $\mathbb{Z}^n$.

Proposition 2. *For any set $\lambda = \{\lambda_i\}$, where $i = 1, \dots, n+1$, $\lambda_1, \dots, \lambda_{n+1} \in N$, and* $\mathrm{GCD}(\lambda_1, \dots, \lambda_{n+1}) = 1$, *there exists a unique (up to transformations from $GL_n(\mathbb{Z})$) simplex $A_1 \cdots A_{n+1}$ such that* (a) $0 = \sum_{i=1}^{n+1} \lambda_i \overline{A}_i$, *and* (b) *the vectors $\overline{A}_1, \dots, \overline{A}_{n+1}$ generate the whole lattice $\mathbb{Z}^n$.*

Proof. Existence. In $\mathbb{Q}^n$ we choose a simplex $B_1 \cdots B_{n+1}$ such that $\sum_{i=1}^{n+1} \lambda_i \overline{B}_i = 0$. The lattice generated by $\overline{B}_1, \dots, \overline{B}_{n+1}$ will be denoted by D. The isomorphism $\varphi\colon D \to \mathbb{Z}^n$ sends the simplex $B_1 \cdots B_{n+1}$ to the desired one.

Uniqueness. Suppose simplices $A_1^{(1)} \cdots A_{n+1}^{(1)}$ and $A_1^{(2)} \cdots A_{n+1}^{(2)}$ both satisfy the condition of the proposition. It follows from (a) that there exists a transformation $\varphi \in GL_n(\mathbb{Q})$ that sends $A_i^{(1)}$ to $A_i^{(2)}$ for all i, $1 \le i \le n+1$. It now follows from (b) that $\varphi \in GL_n(\mathbb{Z})$. The proposition is proved.

Thus to each set λ there corresponds a simplex determined uniquely up to transformations from $GL_n(\mathbb{Z})$. We denote it by S_λ. It is clear that each simplex $A_1 \cdots A_n$ containing O and such that $\overline{A}_1, \dots, \overline{A}_n$ generate $\mathbb{Z}^n$ is S_λ for some λ.

Now we fix a set $\lambda = (\lambda_i)$ and the corresponding simplex $S_\lambda = A_1 \cdots A_{n+1}$. The point $0 = \sum_{i=1}^{n+1} \lambda_i \overline{A}_i$ is contained in that simplex. In the coordinates (x_i) relative to the basis A_{n+1} $(\overline{A}_1 - \overline{A}_{n+1}), \dots, (\overline{A}_n - \overline{A}_{n+1})$ the simplex S_λ can be described by the inequalities $x_i > 0$ $(i = 1, \dots, n)$ and $\sum_{i=1}^{n} x_i < 1$. Let $h = \sum_{i=1}^{n+1} \lambda_i$.

Notice that

$$(\overline{O} - \overline{A}_{n+1}) = \sum_{i=1}^{n} \left(\frac{\lambda_i}{h}\right)(\overline{A}_i - \overline{A}_{n+1}).$$

For $k \in \{1, \dots, h\}$ let

$$O_k = A_{n+1} + \sum_{i=1}^{n} \left\{\frac{\lambda_i}{h} k\right\}(\overline{A}_i - \overline{A}_{n+1}).$$

In particular, $O_1 = 0$ and $O_h = A_{n+1}$. All the O_k are different from each other and lie in the parallelepiped given, in the coordinates (x_i), by the inequalities $0 \le x_i < 1$, $i = 1, \dots, n$. Indeed, if $O_{k_1} = O_{k_2}$, then $k_1\lambda_i \equiv k_2\lambda_i \pmod h$ for all $i = 1, \dots, n$, and therefore also $k_1\lambda_{n+1} \equiv k_2\lambda_{n+1} \pmod h$. Since $\mathrm{GCD}(\lambda_1, \dots, \lambda_{n+1}) = 1$ we have $k_1 \equiv k_2 \pmod h$. Hence $k_1 = k_2$.

Any integral point in the parallelepiped $0 \le x_i < 1$, $i = 1, \dots, n$, is O_l for some l, since the vectors $\overline{A}_i$ generate $\mathbb{Z}^n$.

Let $M_{\varepsilon, n}$ $(0 < \varepsilon < 1)$ denote the set of convex polyhedra P in $\mathbb{Z}^n$ containing O and such that O is the only point of the lattice contained in εP.

Proposition 3. *The following conditions on $\lambda = (\lambda_i)$ are equivalent*:

(a) $S_\lambda \in M_{\varepsilon, n}$.

(b) $O_k \notin \varepsilon S_\lambda$ *for all* $k \in \{2, 3, \dots, h-1\}$.

(c) *For all* $k \in \{2, 3, \dots, h-1\}$, *either* $\sum_{i=1}^{n+1}\{(\lambda_i/h)k\} \in \{2, 3, \dots, n\}$, *or else* $\sum_{i=1}^{n+1}\{(\lambda_i/h)k\} = 1$ *and there exists an* i *such that* $\{(\lambda_i/h)k\} \le (1-\varepsilon)\lambda_i/h$.

Proof. (a) $\Leftrightarrow$ (b). This is obvious. It suffices to remark that of all the points of the lattice only the O_k, $k \in \{2, 3, \dots, h-1\}$ may lie in εS_λ and be different from zero.

(b) $\Leftrightarrow$ (c) Notice that $\sum_{i=1}^{n+1}\{(\lambda_i/h)k\} \in \mathbb{Z}$ for all k, and, if $k \not\equiv 0 \pmod h$, then $\sum_{i=1}^{n+1}\{(\lambda_i/h)k\} \in \{1, 2, \dots, n\}$. If $O_k \in \varepsilon S_\lambda$, then $\sum_{i=1}^{n}\{(\lambda_i/h)k\} < 1$, and therefore $\sum_{i=1}^{n+1}\{(\lambda_i/h)k\} = 1$. Here $\{(\lambda_i/h)k\}$ are the barycentric coordinates of O_k in S_λ and $\{\lambda_i/h\}$ are the barycentric coordinates of O. We assume that $O_k \in \varepsilon S_\lambda$. Then for all $i = 1, \dots, n+1$ we have $\{(\lambda_i/h)k\} > (1-\varepsilon)\lambda_i/h$. Thus (c) implies (b). The converse is equally simple.

§3. Main lemma

Proposition 4. *Suppose* $m, n \in \mathbb{Z}$ *are given, with* $m, n \ge 0$, *and let* $0 < \varepsilon < 1$. *Then there exists a sufficiently large number* $H(n, m, \varepsilon)$ *such that for any* $(n+1)$*-tuple* $(\alpha_1, \dots, \alpha_{n+1}) \in R_+^{n+1}$ *of positive numbers there exists a* $k \in \{2, 3, \dots, H(n, m, \varepsilon)\}$ *such that either* $d_k(\alpha) \notin \{1, 2, \dots, m\}$, *or else* $d_k(\alpha) = 1$ *and, for all* i,

$$\{k\alpha_i\} > (1-\varepsilon)\{\alpha_i\} \quad \text{where } d_k(\alpha) = k\left(1 - \sum_{i=1}^{n+1}\alpha_i\right) + \sum_{i=1}^{n+1}\{k\alpha_i\}.$$

(*It is clear that* $d_k \in \mathbb{Z}$.)

Proof. We use induction on n with m and ε fixed.

Suppose $n = 0$. If $1/(1+\varepsilon) < \alpha < 1$, then even for $k = 2$ we have

$$d_2(\alpha) = 2(1-\alpha) + 2\alpha - 1 = 1 \quad \text{and} \quad \{2\alpha\} = 2\alpha - 1 > \alpha(1-\varepsilon).$$

If $\alpha \ge 1$ then even for $k = 2$ we have $d_2(\alpha) \le 0$.

If $\alpha \le 1/(1+\varepsilon)$, then $d_k(\alpha) \ge k\varepsilon/(1+\varepsilon)$. This means that we can take

$$H(0, m, \varepsilon) = \left[\frac{(m+1)(1+\varepsilon)}{\varepsilon}\right] + 3$$

and take $k = 2$ when $\alpha > 1/(1+\varepsilon)$ and $k = [(m+1)(m+3)/\varepsilon]$ when $\alpha \le 1/(1+\varepsilon)$.

Induction step $(n-1) \to n$. We say that the $(n+1)$-tuple $(\alpha_1, \dots, \alpha_{n+1}) \in R_+^{n+1}$ *satisfies condition* $W_l^{m,\varepsilon}$ if for all $k \in \{2, 3, \dots, l\}$ we have $d_k(\alpha) \in \{1, \dots, m\}$, and if $d_k(\alpha) = 1$ then there exists an i such that $\{k\alpha_i\} \le (1-\varepsilon)\{\alpha_i\}$.

Suppose that for all natural $l > H(n-1, m, \varepsilon)$ there exist $(n+1)$-tuples $(\alpha_1^l, \dots, \alpha_{n+1}^l)$ satisfying conditions $W_l^{m,\varepsilon}$. It is clear that $\alpha_i^l < 1$, because (α_i^l) satisfies $W_2^{m,\varepsilon}$. It follows from the induction hypothesis applied to the $(n+1)$-tuple $(\alpha_1^l, \dots, \alpha_{n+1}^l)$ with α_i^l omitted that $\alpha_i^l \ge 1/H(n-1, m, \varepsilon)$ for all i and l. Indeed, otherwise for $k \le H(n-1, m, \varepsilon)$ we would have

$$\{k\alpha_i^l\} = k\alpha_i^l > (1-\varepsilon)\alpha_i^l$$

and

$$d_k(\alpha_1^l, \dots, \alpha_{n+1}^l) = d_k((\alpha_1^l, \dots, \alpha_{n+1}^l) \text{ with } \alpha_i^l \text{ omitted}).$$

Therefore the $(n+1)$-tuples $(\alpha_i^l, \dots, \alpha_{n+1}^l)$ have an accumulation point in $\mathbb{R}_+^{n+1}$. Let $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ be that point. Replacing the sequence (α_i^l) of $(n+1)$-tuples with a subsequence converging to α, we may assume that $\alpha_i = \lim_{l\to\infty} \alpha_i^l$ for all i.

For all $l > H(n-1, m, \varepsilon)$ we have

$$\left|1 - \sum_{i=1}^{n+1} \alpha_i^l\right| = \left|\frac{1}{l}\left(d_l(\alpha^l) - \sum_{i=1}^{n+1}\{l\alpha_i^l\}\right)\right| \le \frac{1}{l}(m+n+1).$$

Passing to the limit, we get $\sum_{i=1}^{n+1} \alpha_i = 1$.

Now we shall prove that there exists a $k \in \mathbb{N}$, $k \ge 2$, such that $\sum_{i=1}^{n+1}\{k\alpha_i\} = 1$, and for all $i = 1, \dots, n+1$ the inequality $\{k\alpha_i\} > (1-\varepsilon)\alpha_i$ holds. To this end we choose a small $\delta > 0$ such that for all i the equalities $\delta < \varepsilon\alpha_i$, $\delta < 1 - \alpha_i$ and $\delta < 1/(n+1)$ are satisfied. Consider the infinite set of $(n+1)$-tuples $(\{k\alpha_1\}, \dots, \{k\alpha_{n+1}\})$, $k = 1, 2, \dots$. It has an accumulation point, and so there exist k_1 and k_2, $k_1 < k_2$, such that $|\{k_1\alpha_i\} - \{k_2\alpha_i\}| < \delta$ for all i.

It is not difficult to check that $k = 1 + k_2 - k_1$ satisfies all the requirements.

We now have that $d_k(\alpha) = 1$. Since $\{k\alpha_i\} > 0$, the function $d_k(\alpha)$ is continuous in some neighborhood of α. The condition $\{k\alpha_i\} > (1-\varepsilon)\{\alpha_i\}$ for all i is also satisfied in some neighborhood of α. Because d_k takes on integral values, it is equal to 1 in some neighborhood of α and therefore condition $W_k^{m,\varepsilon}$ cannot be satisfied for all α' close enough to α. But this contradicts the choice of α as the limit of α^l. Thus there exists an $l_0 > H(n-1, m, \varepsilon)$ such that for all $\alpha \in R_+^{n+1}$ condition $W_{l_0}^{m,\varepsilon}$ is not satisfied. This l_0 is the desired value for $H(n, m, \varepsilon)$. The proposition is proved.

§4. Arbitrary simplices

Applying Propositions 3 and 4, we have

$$S_\lambda \in M_{\varepsilon,n} \Rightarrow h - 1 < H(n, n, \varepsilon) = N(n, \varepsilon).$$

It follows that there are only finitely many λ such that $S_\lambda \in M_{\varepsilon,n}$.

The case of arbitrary simplices reduces to the case of simplices S_λ by applying Minkowski's lemma on a centrally symmetric convex body. It is easy to see that any simplex can be obtained from a simplex generating the lattice by replacing that lattice with an overlattice of finite index k. For each simplex S_λ there exists $k_{\lambda,\varepsilon}$ such that εS_λ contains a point (different from O) of the overlattice of index k whenever $k > k_{\lambda,\varepsilon}$. Indeed, consider a centrally symmetric (with respect to O) convex body P inscribed in S_λ, and apply Minkowski's lemma to εP and the new overlattice.

Since any simplex $S \in M_{\varepsilon,n}$ can be obtained in that way from the simplex $S_\lambda \in M_{\varepsilon,n}$ and the volume of S does not exceed $k_{\lambda,\varepsilon} \times$ (the volume of S_λ), there exists a constant $c(n, \varepsilon)$ such that no S from $M_{\varepsilon,n}$ has volume greater than $c(n, \varepsilon)$.

From this it follows that, up to the action of $GL_n(\mathbb{Z})$, there are only finitely many simplices in $M_{\varepsilon,n}$.

§5. Conclusion of the proof

Proposition 5. *Up to the action of $GL_n(\mathbb{Z})$, in $\mathbb{Z}^n$ there exist only finitely many convex polyhedra P belonging to $M_{\varepsilon,n}$.*

Proof. *Step* 1. For each line l passing through O let A_l and B_l denote the points of intersection of l and ∂P. Consider the function

$$f(l) = \max\left(\frac{|\overline{A}_l|}{|\overline{B}_l|}, \frac{|\overline{B}_l|}{|\overline{A}_l|}\right).$$

For each point X belonging to ∂P let $\dim X$ denote the dimension of the face on which that point lies. It is clear that $\dim X \in \{0, 1, \dots, n-1\}$.

Step 2. The function $f(l)\colon RP^{n-1} \to R$ is continuous. Therefore there exists a line l such that $f(l)$ is the maximum value. From all such lines we choose one for which the value $\dim A_l + \dim B_l$ is the smallest.

Step 3. Now we shall prove that, with the above choice of the line l,

$$\dim A_l + \dim B_l \le n-1.$$

Indeed, let σ_A be the face containing A_l and σ_B the face containing B_l. If $\dim A_l + \dim B_l \ge n$, then there exists a two-dimensional subspace π containing l such that $\pi \cap \sigma_A \ne A_l$ and $\pi \cap \sigma_B \ne B_l$.

Now we take a closer look at that plane. Let $l_1 = \pi \cap \sigma_A$ and $l_2 = \pi \cap \sigma_B$. If $l_1 \not\parallel l_2$, then in a neighborhood of l there exists a line l' such that $f(l') \ge f(l)$. If $l_1 \| l_2$, then by moving A_l along l_1 and B_l along l_2 we would change l in some way and eventually reach some l' such that $\dim A_{l'} < \dim A_l$ or $\dim B_{l'} < \dim B_l$. In both cases

$$\dim A_{l'} + \dim B_{l'} < \dim A_l + \dim B_l.$$

Step 4. Thus $\dim A_l + \dim B_l \le n-1$. It is easy to see that, although the face σ_A need not be a simplex, we can find some of its vertices $A_1, \dots, A_k$, $k \le \dim A_l + 1$, such that $A_1 \cdots A_k \ni A_l$. Similarly we choose $B_1 \cdots B_m \ni B_l$, $m \le \dim B_l + 1$. An argument similar to that in Step 3 shows that the points $A_1, \dots, A_k$ and $B_1, \dots, B_m$ are in general position and the point O lies in the simplex $A_1 \cdots A_k B_1 \cdots B_m$. Notice also that $k + m \le n+1$. Since for each n there exist only finitely many (up to the action of $GL_n(\mathbb{Z})$) n-dimensional simplices belonging to $M_{\varepsilon, n}$, on taking into account the inequality $k+m \le n+1$ we conclude that $f(l) \le c$, where $c = c(n, \varepsilon)$ is some constant.

Step 5. Now we shall prove that (volume of P) $= V_p \le c_1(n, \varepsilon)$. Indeed, consider the polyhedra P_1 and P_2 obtained from P by the homotheties with center O and coefficients $1/c(n, \varepsilon)$ and $-1/c(n, \varepsilon)$. Then $P_3 = \operatorname{conv}(P_1 \cup P_2) \cdot \varepsilon$ is a centrally symmetric convex body. The polyhedron P_3 is contained in εP, and so one can apply Minkowski's lemma to it. As a result we have $V_{P_3} \le c_2(n)$. But

$$V_P = c^n(n, \varepsilon) V_{P_1} \le \frac{1}{\varepsilon^n} c^n(n, \varepsilon) V_{P_3} \le c_1(n, \varepsilon).$$

Step 6. Thus $V_P \le c_1(n, \varepsilon)$. Now let us take some vertices $A_1, \dots, A_{n+1}$ of that polyhedron which are in general position. It is obvious that $V_{A_1 \cdots A_{n+1}} \le c_1(n, \varepsilon)$.

Up to the action of $GL_n(\mathbb{Z})$, there are only finite many simplices of the form $A_1 \cdots A_{n+1}$. Clearly, for each such simplex there are only finitely many points $x \in \mathbb{Z}^n$ such that

$$V_{\operatorname{conv}(X, A_1, \dots, A_{n+1})} \le c_1(n, E).$$

Indeed, inscribe in $A_1 \cdots A_{n+1}$ a ball of radius r with center at some point Y: $K_Y(r) \subset A_1 \cdots A_{n+1}$. Then

$$V_{\operatorname{conv}(X, A_1 \cdots A_{n+1})} \ge V_{\operatorname{conv}(X, K_Y(r))} \ge c_3(n, r)|XY|,$$

and therefore $|XY| \leq c_4(n, \varepsilon)$. It now follows that X is contained in a bounded domain, and therefore the number of the desired polyhedra is finite.

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