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Corrigendum

# Corrigendum to "Quantum integers and cyclotomy" [J. Number Theory 109 (1) (2004) 120–135]

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ABSTRACT

One of the theorems in the original paper (Borisov, Nathanson, and Wang (2004) [1]) only holds for polynomials and not for rational functions. We explain the mistake, and also prove a new theorem for the rational function case.



Theorem 1 of the paper is incorrect as stated. It is correct in the special case when the functions $f_n(q)$ are polynomials in $q$, and the proof given in the paper is valid in that situation. However for rational functions the proof breaks down, and the statement itself does not hold. The mistake in the proof is the assertion that $f_n(a) = 0$ implies that

$$f_M(a) f_n\left(a^M\right) = f_n(a) f_M\left(a^n\right) = 0.$$

This is true for polynomials, but may not be true for rational functions, if $a^n$ is a root of the denominator of $f_M$. The responsibility for this mistake lies with the first author. The particular partial fix presented below is also due to him.

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In what follows we use the notation and terminology of the original paper [1]. In particular, the *logarithm group* of the number field $K$ is $L(K) = K^*/U$, where $U$ is the torsion of $K^*$ with the group operation written additively. The corresponding map $L : K^* \to L(K)$ satisfies the identity $L(ab) = L(a) + L(b)$.

Instead of Theorem 1, we have the following theorem for rational functions.

**Theorem 1.** *(Corrected.) Let $K$ be an algebraically closed field. Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of rational functions $K(q)$ that satisfies the system of equations $f_{mn}(q) = f_m(q)f_n(q^m)$. Suppose $0$ is not a zero or a pole for any $f_n$. Let $P$ be the set of primes such that* $\operatorname{supp}(\mathcal{F}) = S(P)$. *If* $\operatorname{card}(P) \geq 2$, *then there exist polynomials $r(q)$ and $s(q)$ in $K[q]$ so that*

$$f_n(q) = F_n(q) \cdot \frac{r(q^n)s(q)}{r(q)s(q^n)},$$

*where all zeroes and poles of $F_n$ are roots of unity.*

**Proof.** We will use the following notation.

*Notation.* For a rational function $f(x)$ we denote by $RP(f)$ the set of its zeroes and poles in $K^*$. We denote by $NRP(f)$ the subset of its non-cyclotomic roots and poles, that is those $a \in RP(f)$ that $L(a) \neq 0$.

Because $\operatorname{card}(P) \geq 2$, we can choose positive integers $n$ and $m$ in $\operatorname{supp}(\mathcal{F})$ so that $\log n$ and $\log m$ are rationally linearly independent. The functions $f_n(q)$ and $f_m(q)$ satisfy the functional equation.

$$f_n(q^m)f_m(q) = f_n(q)f_m(q^n).$$

Before proving the theorem, we will prove the following lemma.

**Lemma 1.** *Suppose that, under the notation above, $NRF(f_n)$ is not empty. Then $NRF(f_n) \cap NRF(f_m)$ is not empty.*

**Proof.** Choose $M = m^k$ big enough so that $NRF(f_n(q)) \cap NRF(f_n(q^M))$ is empty. We have

$$\frac{f_M(q)}{f_M(q^n)} = \frac{f_n(q)}{f_n(q^M)}.$$

Suppose $a \in NRP(f_n(q))$. Then either $a \in NRP(f_M(q))$ or $a \in NRP(f_M(q^n))$. In the first case we get some $b = a^{m^i} \in NRP(f_m)$, for $i \geq 0$. In the second case we get $b = a^{n \cdot m^i} \in NRP(f_m)$, for $i \geq 0$.

Doing for $b$ the same thing that we just did for $a$, but switching $n$ and $m$, we get $c \in NRP(f_n)$ which equals $a^{m^u \cdot n^v}$ for some $u, v \geq 0$. This way we can define a map from $NRP(f_n)$ to itself, that send $a$ to $c$ for every choice of $a$. Because the set is finite, this map must have a periodic orbit. This implies that for some $a \in NRP(f_n)$, $a^{m^s \cdot n^t} = a$ for $t, s \geq 0$. Because $a$ is not a root of unity, this implies $m^s n^t - 1 = 0$, so $t = s = 0$. But this means that at the first step of the above construction $b = a$. The lemma is proven. □

When $NRP(f_n) \cap NRP(f_m) \neq \emptyset$, we can change all $f_i$ to $f_i^{(1)}$ as follows. We take $a \in NRP(f_n) \cap NRP(f_m)$ and set

$$f_i^{(1)}(q) = f_i(q) \cdot \left(\frac{q^i - a}{q - a}\right)^{\mathrm{ord}_a(f_n)}.$$

The new sequence $\{f_i^{(1)}\}$ satisfies the same functional equations and $a$ is not in $NRP(f_n^{(1)})$. (It may still belong to $NRP(f_m^{(1)})$.)

To prove the theorem, we are going to be replacing $\{f_i\}$ by $\{f_i^{(1)}\}$ while we can. If at some point we have to stop, then this means that $NRP(f_n) = \emptyset$. This implies that for all $N = n^k$, $NRP(f_N) = \emptyset$. Then for all $i$ and $N$

$$\frac{f_i(q)}{f_i(q^N)} = \frac{f_N(q)}{f_N(q^i)}.$$

The right hand side has only cyclotomic roots and poles. If $f_i(q)$ has a non-cyclotomic root/pole $a$, then for big enough $k$ it is a not a root/pole of $f_i(q^N)$. So all roots and poles of all $f_i$ are roots of unity. This sequence of functions will serve as $F_i(q)$, and we get the result.

So we just need to show that these changes cannot go forever. In order to do this, we introduce a level function on the non-cyclotomic elements of our field. First, we will call two elements $a$ and $b$ $(n, m)$-equivalent if $L(a) = L(b) \cdot n^i \cdot m^j$ for some integer $i$ and $j$. Then for every equivalence class we choose a representative $a$ and define $level(a \cdot n^i \cdot m^j) = i - j$.

Now let us denote by $M$ the biggest difference $level(a) - level(b)$, where $a$ is equivalent to $b$, $a \in NRP(f_n)$ and $b \in NRP(f_m)$. We notice that when we do our moves the level of new roots and poles of $f_m$ is bigger than the level of the one they came from, and the new roots or poles of $f_n$ are of level one less. Consider the tree on the union of sets of roots and poles of $f_n^{(i)}$, $i = 0, 1, 2, \ldots$, by saying that the root/pole $a \in NRP(f_n^{(i)})$ that is involved in the move from $f_n^{(i)}$ to $f_n^{(i+1)}$ is the "parent" of the roots/poles of $f_n^{(i+1)}$ that come from the equation $x^n - a = 0$. Now note that all branches of this tree have length no more than $M + 1$, so the tree is finite. Thus the whole process is finite, and the theorem is proven. □

## References


[1] A. Borisov, M.B. Nathanson, Y. Wang, Quantum integers and cyclotomy, J. Number Theory 109 (1) (2004) 120–135.