

ON THE LOG DISCREPANCIES IN MORI CONTRACTIONS

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ABSTRACT. It was conjectured by M^cKernan and Shokurov that for all Mori contractions from X to Y of given dimensions, for any positive ε there is a positive δ such that if X is ε -log terminal, then Y is δ -log terminal. We prove this conjecture in the toric case and discuss the dependence of δ on ε , which seems mysterious.

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1. Introduction

The main subject of this paper is the following 2003 conjecture of James M^cKernan:

Conjecture 1 (M^cKernan). *For fixed positive integers m, n and a real number $\varepsilon > 0$ there exists a $\delta = \delta_{m,n}(\varepsilon) > 0$ such that the following holds: Let X be a \mathbb{Q} -factorial variety, and $f: X \rightarrow Y$ be a Mori fiber space with $\dim Y = n$, $\dim X = m + n$. Assume that X is ε -log terminal. Then Y is δ -log terminal.*

A related stronger conjecture was suggested by V.V. Shokurov. Let $f: X \rightarrow Y$ be a proper surjective morphism with connected fibers of normal varieties, so that X/Y is of relative Fano type (see definitions below) and let Δ be a \mathbb{Q} -divisor on X such that $K_X + \Delta = f^*L$ for some \mathbb{Q} -divisor L on Y .

By Kawamata's subadjunction formula [Kaw97, Kaw98], see also [Amb05], one has $K_X + \Delta = f^*(K_Y + R + B)$, where R is the discriminant part, and B is the "moduli" part, a \mathbb{Q} -divisor defined only up to \mathbb{Q} -linear equivalence.

Conjecture 2 (Shokurov). *In the above settings, assume that (X, Δ) is ε -log terminal. Then there exists $\delta = \delta_{m,n}(\varepsilon) > 0$ and an effective moduli part B , such that $(Y, R + B)$ is δ -log terminal.*

Conjecture 2 clearly implies Conjecture 1: for a Mori fiber space consider a large integer $N \gg 0$ and a generic element D of a very ample linear system $-NK_X + f^*M$ for some M on Y , and let $\Delta = \frac{1}{N}D$. Then $K_X + \Delta = f^*L$ and for the minimal log discrepancies one has

$$\text{mld}(X, \Delta) = \text{mld}(X) + \frac{1}{N} \quad \text{and} \quad \text{mld}(Y, R + B) \leq \text{mld}(Y).$$

Taking the limit $N \rightarrow \infty$ gives the implication.

We refer the reader to [KM98] for basic definitions and results of the Minimal Model Program, some of which we briefly recall below. For any normal variety X for which some positive multiple of the canonical class K_X is Cartier, one defines *discrepancies* $a_i \in \mathbb{Q}$ by the formula

$$K_{X'} = \pi^* K_X + \sum a_i E_i, \quad \text{Exc}(\pi) = \cup E_i,$$

in which $\pi: X' \rightarrow X$ is a resolution of singularities, and E_i are the irreducible exceptional divisors of π . The *log discrepancies* are the numbers $a_i^{\log} = a_i + 1$. The *minimal log discrepancy* $\text{mld}(X)$ is the minimum of log discrepancies, going over all resolutions of singularities (equivalently for one resolution $X' \rightarrow X$ such that $\text{Exc}(\pi)$ is a normal crossing divisor).

A variety is said to be ε -log terminal (abbreviated below to ε -lt) if its log discrepancies are $> \varepsilon$, i.e. if for ordinary discrepancies one has $a_i > -1 + \varepsilon$. Similarly, a variety is ε -log canonical if the log discrepancies are $\geq \varepsilon$. In particular, 0-log terminal is the same as Kawamata log terminal (klt), and 0-log canonical is the same as log canonical.

We recall that $f: X \rightarrow Y$ is a *Mori fiber space* if f is projective, $-K_X$ is f -ample, and the relative Picard number is $\rho(X/Y) = 1$. The assumption that X is \mathbb{Q} -factorial implies that so is Y (cf. [KMM87, Lemma 5-1-5]).

Finally, a variety X is called a variety of Fano type (FT) if there exists an effective \mathbb{Q} -divisor D such that the pair (X, D) is klt and $-(K_X + D)$ is nef and big.

There are numerous motivations for the above conjectures. The case $\varepsilon = \delta = 0$ of Conjecture 1, i.e. “ X is klt implies Y is klt” follows easily by cutting X with m general hyperplanes and reducing to a finite surjective morphism. Even if X is *not* \mathbb{Q} -factorial, the implication “ X is klt implies (Y, Δ) is klt for an appropriate divisor Δ ” is true, as proved by Fujino [Fuj99].

The first nontrivial case with $\varepsilon > 0$ appears when $\dim X = 3$ and $\dim Y = 2$, i.e. when $f: X \rightarrow Y$ is a singular conic bundle. Mori and Prokhorov [MP08] considered the case when X is terminal. In this case, they proved Iskovskikh conjecture which says that Y must have at worst Du Val singularities. This proves that one can take $\delta_{1,2}(1) = 1 - c$ for any $c > 0$. Yuri Prokhorov also showed us several examples of conic bundles of the form $(\mathbb{P}^1 \times \mathbb{A}^2)/G \rightarrow \mathbb{A}^2/G$ for a cyclic group G which indicate that Conjecture 1 is plausible.

Conjecture 1 may also be viewed as the local analogue of Borisov-Alexeev-Borisov (BAB) boundedness conjecture [BB92, Ale94] which says that for fixed n and $\varepsilon > 0$ the family of n -dimensional ε -lt Fano varieties is bounded.

Indeed, if X happen to be Fano varieties, then the family of possible ε -lt varieties X is bounded by the BAB conjecture. Then the family of possible varieties Y must be bounded, so some $\delta(\varepsilon) > 0$ must exist. Vice versa, when trying to prove BAB conjecture by induction, Conjecture 1 naturally appears as one of the steps. In this sense, it can be considered to be “the local BAB conjecture”.

The main result of the present paper is the following

Theorem 1. *Conjecture 1 holds in the toric case, i.e. when $f: X \rightarrow Y$ is a morphism of toric varieties corresponding to a map of fans $(N_x, \Sigma_X) \rightarrow (N_Y, \Sigma_Y)$.*

Note that in the toric case, if one denotes by Δ the sum of torus invariant divisors with coefficients 1, then one has $K_X + \Delta = 0$ and the pair (X, Δ) is log canonical with $\text{mld}(X, \Delta) = 0$. Thus, the more general Conjecture 2 does not fit the toric case very well.

A very interesting question is to find the asymptotic of the function $\delta(\varepsilon)$ as $\varepsilon \rightarrow 0$. Concerning this, we prove the following:

Theorem 2. *In the conditions of Theorem 1, suppose additionally that the generic fiber of f is a finite, unramified in codimension one, toric quotient of a fixed toric Fano variety P . Then there exists a constant C such that $\delta \geq C \cdot \varepsilon^{m+1}$.*

On the other hand, we prove:

Theorem 3. *There exist a sequence of toric Mori fiber spaces with $m = n = 2$ such that $\text{mld}(X) \rightarrow 0$ and $\text{mld}(Y) \approx C \cdot \text{mld}(X)^4$.*

Acknowledgments. This research originated in a discussion at the ACC workshop at the American Institute of Mathematics in May of 2012, as an attempt to answer a question brought up by Yuri Prokhorov. The authors wish to thank Yu. Prokhorov, V.V. Shokurov and other participants of this workshop for many fruitful discussions.

2. Proofs of the main results

We continue with the notation of the Introduction. We first need to examine the combinatorics of the fans of X and Y . We refer to [Ful93] or [Oda88] for the general theory of toric varieties. We work over \mathbb{C} for simplicity, although, as usual in toric geometry, the results remain true over a field of positive characteristic as well.

Recall that a toric variety X is given by a pair (N_X, Σ_X) where N_X is a lattice (called the lattice of valuations) and Σ_X is a rational polyhedral fan in $N_X \otimes \mathbb{R}$. A toric map from a toric variety X to a toric variety Y is given by a linear map $F: N_X \rightarrow N_Y$ such that its extension $F_{\mathbb{R}}: N_X \otimes \mathbb{R} \rightarrow N_Y \otimes \mathbb{R}$ sends every cone in the fan Σ_X to a cone in the fan Σ_Y .

We denote by N_Z the lattice $\text{Ker}(F)$, and by Σ_Z the restriction of Σ_X to $\text{Ker}(F_{\mathbb{R}})$. We recall the following basic facts:

Fact 1. *The morphism $f: X \rightarrow Y$ is proper iff $F_{\mathbb{R}}^{-1}(\text{Supp } \Sigma_Y) = \text{Supp } \Sigma_X$.*

Fact 2. *A general fiber of $f: X \rightarrow Y$ is a product of a torus of dimension $\dim N_Z$ with the finite part, the product of finitely many copies of the group schemes $\mu_{r_i} = \text{Spec } k[z]/(z^{r_i} - 1)$. The character group of the finite part is the torsion subgroup of $\text{coker}(F: N_X \rightarrow N_Y)$.*

A Mori fiber space $f: X \rightarrow Y$ is a surjective proper morphism with connected fibers, and a general fiber is connected and reduced. Therefore, in our situation one has $F_{\mathbb{R}}^{-1}(\text{Supp } \Sigma_Y) = \text{Supp } \Sigma_X$, and the morphism of lattices $F: N_X \rightarrow N_Y$ is surjective.

Fact 3. *A toric variety X is \mathbb{Q} -Gorenstein, i.e. the canonical divisor K_X is \mathbb{Q} -Cartier iff there exists a function $\ell = \ell_{-K_X}: \text{Supp } \Sigma_X \rightarrow \mathbb{R}$ which is linear on each cone $\sigma \in \Sigma_X$ and such that $\ell(P_i) = 1$ for each shortest integral generator P_i of each ray R_i of σ .*

Fact 4. *A toric variety X is \mathbb{Q} -factorial, i.e. every Weil divisor is \mathbb{Q} -Cartier iff the fan Σ is simplicial, i.e. every cone is a simplex.*

Fact 5. *The $\text{mld}(X)$ is computed as the minimum of the piece-wise function $\ell = \ell_{-K_X}$ on $\text{Supp } \Sigma_X \cap N_X \setminus \{0\}$.*

Obviously, our problem is local on Y , so we can assume that Y is affine. Since X and Y are \mathbb{Q} -factorial, Y is a quotient of A^n by a finite abelian group. Combinatorially, it is obtained from the standard cone $C = \{(x_1, \dots, x_n | \forall x_i \geq 0\}$ in \mathbb{R}^n , with the lattice being a finite extension of the standard lattice \mathbb{Z}^n . Thus, the fan Σ_Y consists of the cone C and its faces.

If the shortest integral generators of the cone C are the standard basis vectors e_i then the linear function computing $\text{mld}(Y)$ is simply $\ell_{-K_Y} = \sum x_i$.

Proposition 4. *Suppose that $f : X \rightarrow Y$ is a toric Mori fiber space, with \mathbb{Q} -factorial X and affine Y as above. Denote by $F : N_X \rightarrow N_Y$ the map of the corresponding lattices, and extend it to the linear map $F_{\mathbb{R}}$ from $N_X \otimes \mathbb{R} = \mathbb{R}^{n+m}$ to $N_Y \otimes \mathbb{R} = \mathbb{R}^n$. We choose the basis of $N_X \otimes \mathbb{R}$ so that the map $F_{\mathbb{R}}$ is the projection of \mathbb{R}^{n+m} to the last n coordinates. Then the following is true about the fan of X in $N_X \otimes \mathbb{R}$.*

1) *It has exactly $(n + m + 1)$ one-dimensional cones (rays) R_i , $i = 0, 1, \dots, m + n$ of which R_0, \dots, R_m are in $\text{Ker}(F_{\mathbb{R}})$ and for all $i = m + 1, \dots, m + n$ the ray R_i is mapped by $F_{\mathbb{R}}$ to the ray of C , which is the span of the standard basis vector e_{i-m} .*

2) *Any generators of R_i for $i = 0, \dots, m$, form a simplex that contains 0.*

3) *The cones in the fan of X are precisely the simplicial cones generated by R_i for $i \in S \subset \{0, 1, \dots, n + m\}$, where S does not contain $\{0, 1, \dots, m\}$.*

Proof. The condition $F_{\mathbb{R}}^{-1}(C) = \text{Supp } \Sigma_X$ implies that

- (1) The fan Σ_Z is complete, i.e. $\text{Supp } \Sigma_Z = N_Z \otimes \mathbb{R}$. In particular, Σ_X has at least $m + 1$ rays in $\text{ker } F_{\mathbb{R}}$.
- (2) For each of the n rays of C , there exists at least one ray of Σ_X lying over it.

Recall that the Picard group of \mathbb{R} -Cartier divisors on a toric variety X is the quotient of the space of piece-wise linear functions modulo the space of linear functions on Σ_X . Since both fans are simplicial (because X, Y are both \mathbb{Q} -factorial) and full-dimensional, the relative Picard number $\rho(X/Y)$ is the difference between the number of rays of X and Y minus the relative dimension, m .

Therefore, Σ_X has $m + n + 1$ rays. Thus, Σ_X has no other rays other than the $(m + 1) + n$ rays listed above, and over each ray of C there exists a *unique* ray of Σ_X . This proves (1).

Finally, for this set of $n + m + 1$ rays there is only one simplicial fan with support $F_{\mathbb{R}}^{-1}(C)$: the one described in (3). This proves (2) and (3). \square

We now choose a basis in $N_X \otimes \mathbb{R}$ so that the last n coordinate vectors are the primitive elements of N_X on the rays R_i , $i \geq m + 1$, denoted by P_i . For $i = 0, \dots, m$ we also denote by P_i the primitive elements of $N_Z = \text{ker}(N_X \rightarrow N_Y)$ on the rays R_i .

By the above Proposition, the fan Σ_X is isomorphic to the Cartesian product $\Sigma_Y \times \Sigma_Z$. Since a general fiber of $X \rightarrow Y$ is connected, the map $F : N_X \rightarrow N_Y$ is surjective. Therefore, one has $N_X \simeq N_Y \times N_Z$. However, one need *not* have $(N_X, \Sigma_X) \simeq (N_Y, \Sigma_Y) \times (N_Z, \Sigma_Z)$. In particular, it is possible that $F(P_i)$ are not primitive in the lattice N_Y .

Denote by Δ the simplex with vertices P_i in $\text{Ker}(F)$. This structure defines the toric Fano variety Z of Picard number one, which is the generic fiber of f . We choose the coordinates in $N_X \otimes \mathbb{R} = \mathbb{R}^{n+m}$ so that the lattice generated by P_i is the standard $\mathbb{Z}^m \subset \mathbb{R}^m = \text{Ker}(F_{\mathbb{R}})$; the lattice N_Z is a finite extension of it.

We now describe our basic strategy for the proof of Theorem 1. Recall that $\text{mld}(Y)$ of a toric variety Y is computed as the minimum of the linear function $\sum_{i=1}^n x_i$ over the non-zero points of $N_Y \cap C$.

According to Fact 5, the mld of a toric singularity is the minimum of the log discrepancies of the non-zero points of the corresponding cone, where the log discrepancy of a point is the value on it of the linear function that equals 1 on the rays of the cone. Suppose that for Y this minimum is achieved at some point A . We want to prove that if the log discrepancy of A is very small, there must exist a point in N_X , in one of the cones of Σ_X , for which the log discrepancy is also small (less than the given ε). To look for this point, we take a preimage P of A in N_X (it is possible, because $N_X \rightarrow N_Y$ is surjective), and consider its multiples $P, 2P, \dots, tP$ modulo the lattice \mathbb{Z}^{n+m} , for some t to be specified later. If the log discrepancy of A is really small, then we can choose a fairly large t such that for all these points the sum of the last n coordinates is still small. By Dirichlet Box Principle, we can choose two of these points to be close to each other, and take their difference. If we subtract in the correct order, this produces a point Q in N_X with the last n coordinates nonnegative and with small sum; and the projection to the first m coordinates being near the origin. Because the union of the cones for Z is the whole \mathbb{R}^m , this projection must belong to some cone, which implies that Q lies in some cone for X and has a small log discrepancy there.

To illustrate the method, we first establish Theorem 1 in the particular case when Z is an unramified in codimension one quotient of the usual projective space \mathbb{P}^m . This means that the barycentric coordinates of 0 in the simplex Δ are $(\frac{1}{m+1}, \dots, \frac{1}{m+1})$.

Proposition 5. *In the above notation, suppose additionally that the points P_i for $1 \leq i \leq m$ are the standard $e_i \in \mathbb{R}^{n+m}$, and $P_0 = (-1, \dots, -1; 0, \dots, 0)$ (Here the semicolon separates the first m coordinates from the last n). Then for any $\varepsilon > 0$, if $\text{mld}(X) > \varepsilon$, then $\text{mld}(Y) > \delta = (\frac{\varepsilon}{2m})^{m+1}$.*

Proof. Suppose that $\text{mld}(Y) \leq \delta$. Denote the point in N_Y on which the mld is achieved, by A . In other words, $A = (a_1, \dots, a_n)$, where a_i are nonnegative, not all zero, and $\sum a_i \leq \delta$. Because F is surjective, $A = F(P)$ for some $P \in N_X$. Suppose $P = (b_1, \dots, b_m; a_1, \dots, a_n)$. We may additionally assume that all b_i are in $[0, 1)$, because $\mathbb{Z}^m \subseteq N_Z$.

Choose $t = \delta^{-\frac{m}{m+1}}$. For all integers $k \in [0, t)$ consider the points $P_k = kP \bmod \mathbb{Z}^{n+m} = ([kb_1], \dots, [kb_m]; ka_1, \dots, ka_n)$ and their projections to \mathbb{R}^m : $\bar{P}_k = ([kb_1], \dots, [kb_m])$.

Lemma 6. *Suppose for all integer $k \in [0, t)$, $Q_k = (b_{1,k}, \dots, b_{m,k})$ are arbitrary points in $[0, 1)^m$. Then there exist i and j so that for all $l = 1, \dots, m$ we have $|b_{l,i} - b_{l,j}| \leq t^{-1/m}$.*

Proof of Lemma. Identify $[0, 1)^m$ with the quotient $\mathbb{R}^m/\mathbb{Z}^m$, with the usual Haar probability measure. For each Q_i consider a closed box neighborhood of it defined

by the conditions $x_l \in [b_{l,i} - \frac{1}{2}t^{-1/m}, b_i + \frac{1}{2}t^{-1/m}] \pmod{Z}$. The volume (i.e. the Haar measure) of each such box is t^{-1} . Note that the total number of points is $\lfloor t \rfloor > t$, so the total sum of the volumes is greater than 1. Thus there exist i and j such that the corresponding boxes intersect. The triangle inequality in \mathbb{R}/\mathbb{Z} implies the result. \square

We apply the above Lemma to the points $Q_k = \bar{P}_k$. Without loss of generality, we can assume that $i < j$. Consider the point $Q = P_j - P_i \in N_X$. In coordinates, $Q = (\lfloor jb_1 \rfloor - \lfloor ib_1 \rfloor, \dots, \lfloor jb_m \rfloor - \lfloor ib_m \rfloor; (j-i)a_1, \dots, (j-i)a_n)$. Note that $0 < j-i \leq t$, so the sum of the last n coordinates of Q is at most $t\delta$. Suppose that for $l \geq m+1$ we have $F(P_l) = c_l \cdot e_{l-m}$. Then the contribution to the log discrepancy of Q from the last n coordinates is

$$k \sum_{l=m+1}^{n+m} \frac{a_{l-m}}{c_l} \leq k(a_l + \dots + a_n) \leq k \cdot \delta.$$

The first m coordinates of Q are less than $t^{-1/m}$ in absolute value. Denote by \bar{Q} the natural projection of Q to $N_Z \otimes \mathbb{R}$:

$$\bar{Q} = (\lfloor jb_1 \rfloor - \lfloor ib_1 \rfloor, \dots, \lfloor jb_m \rfloor - \lfloor ib_m \rfloor) = (q_1, \dots, q_m).$$

Then \bar{Q} belongs to one of the cones of the fan for Z as follows.

Case 1. All q_l are nonnegative. Then \bar{Q} belongs to the cone $x_i \geq 0$, which is the span of P_i , for $1 \leq i \leq m$. The contribution to the log discrepancy from the first m coordinates is at most $m \cdot t^{-1/m}$.

Case 2. At least one of the numbers q_l is negative. Without loss of generality, we can assume that q_1 is the smallest (i.e. the most negative) of q_l . Then \bar{Q} lies in the span of $P_0; P_2, \dots, P_m$. Its coordinates in that basis are $(-q_1; q_2 - q_1, \dots, q_m - q_1)$. The contribution to the log discrepancy from the first m coordinates is at most $(2m-1) \cdot t^{-1/m}$.

Putting it together, the log discrepancy of Q is at most $(2m-1)t^{-1/m} + t\delta$. Since we chose $t = \delta^{-\frac{m}{m+1}}$, we get the log discrepancy of Q to be at most $2m\delta^{\frac{1}{m+1}} \leq \varepsilon$, which contradicts $\text{mld}(X) > \varepsilon$.

This completes the proof of Proposition 5. \square

Remark 7. *One can improve the above estimate slightly by choosing t to be a suitable constant times $\delta^{-\frac{1}{m+1}}$, and by a more “projectively symmetric” estimates for \bar{Q} . But it will still give the result of the form $\delta \geq \text{const}(m) \cdot \varepsilon^{m+1}$, and would make the exposition considerably more muddled.*

Proof of Theorems 1 and 2. A slight generalization of the above argument yields Theorem 2. Indeed, suppose Z is an arbitrary toric Fano variety of dimension m with the corresponding simplex Δ , and suppose that the barycentric coordinates of 0 in Δ are y_1, y_2, \dots, y_{m+1} . We can fix the vertices $P_i, i \leq m+1$, in \mathbb{Z}^m . As before, we can take t to be $\delta^{-\frac{1}{m+1}}$. We apply the same Lemma (though one can get a somewhat better estimate by generalizing scaling the boxes, keeping the same volume). As a result, the absolute values of all coordinates of the point \bar{Q} are again at most $t^{-\frac{1}{m}} = \delta^{\frac{1}{m+1}}$. So for each of the $(m+1)$ linear functions corresponding to the m -dimensional cones of the fan for Z , the log discrepancy for \bar{Q} will be bounded by constant multiple of $\delta^{\frac{1}{m+1}}$. The same estimate as above proves that for the fixed y_1, y_2, \dots, y_{m+1} one can choose $\delta = \text{const} \cdot \varepsilon^{m+1}$, thus proving Theorem 2.

Finally, Theorem 1 follows from Theorem 2 by a simple observation that if $\text{mld}(X) > \varepsilon$, then also $\text{mld}(Z) > \varepsilon$. By the main result of [BB92] (BAB Conjecture for toric varieties) there are only finitely many possible Fano varieties Z with $\text{mld}(Z) > \varepsilon$. \square

While the above argument may seem to imply the existence of a general estimate for the $\text{mld}(Y)$ in terms of $\text{mld}(X)$ in the form $\text{const}(m) \cdot \varepsilon^{m+1}$, the constant depends implicitly on ε . In fact, one simply cannot hope for the estimate above, in light of the following example which proves Theorem 3.

Example 8. We fix $n = m = 2$. Suppose l is a natural number. Consider a triangle in \mathbb{Z}^2 with vertices $(1, 0), (-(l-1), 1), (-(l-1), -1)$. This gives a weighted projective space; we multiply it by A^2 , and consider the quotient by the group μ_r , where $r = l^4 + 1$, given by the weights $\frac{1}{r}(l, l^2; 1, 1)$. In other words, we take a lattice $\mathbb{Z}^2 \subset \mathbb{Z}^4$, and enlarge the latter by adjoining the point $\frac{1}{r}(l, l^2; 1, 1)$. The rays are $(1, 0, 0, 0), (-(l-1), 1, 0, 0), (-(l-1), -1, 0, 0); (0, 0, 1, 0), (0, 0, 0, 1)$. The map F is just the projection to the last two coordinates. The variety Y is a cyclic quotient singularity of type $\frac{1}{r}(1, 1)$.

We claim that for the above Example the $\text{mld}(Y)$ is asymptotically $\frac{1}{l^4}$, while $\text{mld}(X)$ is asymptotically at least $\frac{1}{2l}$. This would obviously imply Theorem 3.

The first part is easy: $\text{mld}(Y) = 2/r$, which is asymptotically $2/l^4$.

For the estimate on $\text{mld}(X)$, consider the point $N = \frac{1}{r}(l, l^2; 1, 1)$ in \mathbb{R}^4 . We need to prove that no sums kN and points of Z^4 have small log discrepancy, in any of the cones of X . Consider such point $Q = kP + B$, where k is an integer from 1 to $r-1$ and $B \in \mathbb{Z}^4$. Clearly, we can assume that the last two coordinates of B are zero, thus $B \in \mathbb{Z}^2 \subset \mathbb{Z}^4$. Note the following.

1) If $k > (l^3)/2$, then the contribution from the last two coordinates is already too big. So we are only concerned with $k \leq (l^3)/2$.

2) Since $k \leq (l^3)/2$, the first coordinate in kN is between 0 and $1/2$. Therefore the points in the left cone are of no concern: they would have log discrepancy contribution from the first two coordinates at least $(1/2)/(l-1)$. For the points in the upper or lower cone, if $k > l^2/2$, then the log discrepancy is at least $kl/r > (l^3)/(2r)$, which is about $1/(2l)$. So we only need to consider $k \leq l^2/2$.

3) Since $k \leq l^2/2$, the second coordinate of kN is between 0 and about $1/2$. This rules out points in the lower cone. For the upper cone, we clearly only need to be concerned with the points $kN + (0, 0)$. And there the smallest $(x_1 + lx_2)$ value is at least $1 \times l^2/r = l^3/r$, which is about $1/l$.

3. Miscellaneous remarks

It may seem like one cannot avoid using the BAB conjecture to prove Theorem 1. However, there is an explicit version of the toric BAB theorem (proved by Lagarias and Ziegler [LZ91], and originally by Hensley [Hen83], before [BB92]) which may probably be used to get an explicit bound of the form $\delta = C(m)\varepsilon^{d(m)}$. However this is by no means automatic, and the correct power $d(m)$ is highly mysterious. Probably, for $m = 2$ it is 4, but in higher dimensions the answer is not obvious.

By a more careful generalization of the argument for \mathbb{P}^m , one can get an estimate for δ in terms of m, ε and the Tian's alpha invariant of Z (that essentially measures

“asymmetry” of the simplex Δ). Perhaps a generalization of this argument to non-toric case will naturally use this invariant as well.

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