BOUNDEDNESS THEOREM FOR FANO LOG-THREEFOLDS

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1. Introduction

First of all, let me recall necessary definitions and list some known results and conjectures in direction of boundedness of Fano manifolds. All varieties in this paper are over the field of complex numbers.

Definition 1.1. *A normal variety X is called the variety with log-terminal singularities if mK_X is a Cartier divisor for some integer m and there exists a resolution $\pi : Y \longrightarrow X$ of singularities of X such that exceptional divisors F_i of π have simple normal crossings and in formula $K_Y = \pi^* K_X + \sum(a_i F_i)$ all $a_i > -1$*

Definition 1.2. *The index (or Gorenstein index) of a variety X is a minimal natural number m, s.t. mK_X is a Cartier divisor. Of course, indices are defined for $\mathbb{Q}$-Gorenstein varieties only.*

Definition 1.3. *A three-dimensional algebraic variety X is called Fano log-threefold if the following conditions hold.*

1. *X has log-terminal singularities,*
2. *X is $\mathbb{Q}$ -factorial,*
3. *Picard number $\rho(X) = 1$,*
4. *$-K_X$ is ample.*

Remark 1.1. *This is only my terminology inspired by the term "$\mathbb{Q}$ -Fano threefolds".*

The following statement will be proven in this paper.

Main Theorem.

For an arbitrary natural n all Fano log-threefolds of index n lie in finite number of families.

Remark 1.2. *Unfortunately, no effective bounds on any invariants of X will be given because of Noetherian induction in the section 4.*

Here are some results in the direction of boundedness of Fano manifolds.

1) Boundedness theorem for smooth Fano manifolds of an arbitrary dimension is proven by Kollár, Miyaoka and Mori in [13]. Before this result there were several proofs with extra condition $\rho(X) = 1$. Three-dimensional smooth case was also treated before by a long work of many authors beginning with Fano himself. See [7] for discussion.

2)Two-dimensional Fano varieties are traditionally called Del Pezzo surfaces. Smooth (=terminal) case is fairly easy and the answer is the following.

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$P^1 \times P^1, P^2$ with 0 to 8 blown-up points in general position. (The generality of position may be stated precisely.)

I should notice here that there are many difficult problems concerning Del Pezzo surfaces if basic field is NOT algebraically closed.

Log-terminal case (with arbitrary Picard number) was studied by Alexeev and Nikulin (see[17]). One of the main results in this direction is boundedness under the condition of bounded multiplicity of singularities. Let me mention that by using methods of this paper one can obtain a new simple proof of some intermediate result, namely boundedness under the condition of bounded index.

3)The model case of toric varieties of arbitrary dimension is treated in [5], see also [3].

4) Boundedness of Fano threefolds with $\rho = 1$ and with terminal singularities is proven by Kawamata in [8].

5) Boundedness of Fano threefolds with terminal singularities with no extra conditions is announced by Mori (joint work with Kollár and Miyaoka, not published as of August 1995).

All these results justify the following conjecture.

Conjecture 1.1. *The family of all Fano varieties of a given dimension with discrepancies of singularities greater than or equal to $-1+\epsilon$, where ϵ is an arbitrary given positive real number, is bounded.*

Remark 1.3. *This conjecture is so natural that probably many people suspected it but I didn't see it published. Batyrev proposed the weaker variant of this conjecture, where the condition on discrepancies is replaced by the condition of boundedness of index. ([4]) Recently Alexeev told me that he also stated the above conjecture as a part of a general phenomenon noticed by Shokurov that some geometric invariants (in this case minimal discrepancies of Fano varieties) can accumulate only from above (below). See [2] for a discussion.*

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2. Preliminary remarks and first lemmas

In [12] Kollár proved that all three-dimensional normal varieties X with an ample Cartier divisor D lie in finite number of families if two higher coefficients of Hilbert polynomial $P(m) = \chi(mD)$ are bounded. In our case of three-dimensional Fano varieties of index n it works as follows. Let D be equal to $-nK_X$. Then it is a Cartier divisor and it follows from general theory of Riemann-Roch that

$\chi(O_X(-mnK_X)) = \frac{1}{12}(-K_X)^3 nm(nm+1)(2nm+1) + \alpha m + \beta$, where α and β are some constants depending on X.

Therefore in order to prove the Main Theorem we only need to prove that $(-K_X)^3$ is bounded. The following lemma shows that in our case it is also equivalent to the condition that $h^0(-2nK_X)$ is bounded.

Lemma 2.1. *For an arbitrary Fano log-threefold X of index n (actually, only conditions (1) and (3) are used) the following inequality holds.*

$h^0(-2nK_X) \geq (-K_X)^3(n^3 + \frac{1}{2}n^2) - 1$

Proof. By the Kawamata-Vieweg vanishing theorem $h^i(-mnK_X) = 0$ for $i > 0, m \geq 0$. Therefore $h^o(-mnK_X) = \frac{1}{12}(-K_X)^3 nm(nm+1)(2nm+1) + \alpha m + \beta$ for $m \geq 0$. Let us consider "the second derivative at 1".

$h^o(-2nK_X) - 2h^o(-nK_X) + h^0(O_X) = (-K_X)^3(n^3 + \frac{1}{2}n^2)$.

Now the desired inequality follows directly from the facts that $h^0(O_X) = 1$ and $h^0(-nK_X) \geq 0$.

Lemma 2.2. *Suppose $v \in V$ is a closed point of a k-dimensional variety with multiplicity of its local ring r, and D is a nef and big $\mathbb{Q}$ -Cartier divisor on V. Suppose further that the general point x of V can be connected to v by some irreducible curve γ_x, such that $\gamma_x \cdot D \leq d$. Then $D^k \leq r \cdot d^k$.*

Proof. Since D is nef and big, for sufficiently large m such that mD is a Cartier divisor one has that $h^o(O_V(mD)) = \frac{m^k D^k}{k!} + O(m^{k-1})$. Therefore if $D^k > r \cdot d^k$ then for $m >> 0$ one can find a non-zero global section $s \in H^0(O_V(mD))$ such that its image by trivialization map of $O_V(mD)$ in v lies in $(md+1)$-th power of the maximal ideal of v. Then every curve γ_x lies in $Supp(s)$, which is impossible.

Remark 2.1. *The above lemma is very general. In applications V will be our Fano log-threefold X and D will be $(-K_X)$.*

3. Covering family and first division into cases

Remark 3.1. *(about notation). First of all, all curves in this paper are reduced and irreducible as opposed to the chain of curves which is a one-dimensional reduced and connected but usually reducible scheme. Whenever we have a family of curves, a general element is reduced and irreducible unless the opposite is explicitly specified. Second, we will often consider birational varieties. Doing this we will usually identify curves on different varieties if they coincide in their general points. Namely, let $X \leftarrow - \rightarrow X'$ and $L \subset X, L' \subset X'$ be curves. Then L and L' are identified if there are Zariski open subsets $U \subset X$ and $U' \subset X'$, such that the above rational map is defined on them and $U \cong U'$, $L \cap U \cong L' \cap U' \neq \emptyset$ via it. The identified curves will be usually denoted by the same symbol. The same convention will be used for two-dimensional subvarieties. If it is necessary to point out that, say, a prime divisor S is considered on variety X it will be denoted by S_X. Another convention is that $\{l\}$ will denote the family of curves with general element l and $\{H\}$ will denote the LINEAR system of Weil divisors with general element H. It will be clear in every particular case why these conventions agree with each other.*

Now we start to prove our Main Theorem. Suppose X is a Fano log-threefold, $\pi_X^Y : Y \longrightarrow X$ is its $\mathbb{Q}$-factorial terminal modification, $\pi_Y^{Y_1} : Y_1 \longrightarrow Y$ is a resolution of isolated singularities of Y. By the Miyaoka-Mori theorem ([15], see also [9]) there exists a covering family of rational curves $\{l\}$, such that $l \cdot (-K_X) \leq 6$. The family $\{l\}$ is free on Y_1 that is a small full deformation of l covers a small neighborhood of l. (See [16].) We can and will denote by $\{l_{Y_1}\}$ the full family, that is (some Zariski open subset of) a component of the scheme of morphisms from P^1 to Y_1. Consider the RC-fibration $\phi : Y_1 - - \rightarrow Z$, associated with $\{l\}$. (See [13], [14].) The following cases are possible.

(0) $dimZ = 0$. In [14] such X are called primitive. It implies that two general points of Y_1 can be joined by chain of no more than 3 curves from $\{l\}$. It follows

now from one of the "gluing lemmas" ([14]) that we can glue them together and obtain new family $\{l'\}$. Then we can apply to it lemma 2.2 and obtain that $(-K_X)^3 \leq (3 \cdot l \cdot (-K_X))^3 \leq (3 \cdot 6)^3$.

(1) $dimZ = 1$. In this case after some additional blowing-up $\tilde{Y} \longrightarrow Y_1$ we obtain a morphism $\phi_{\tilde{Y}} : \tilde{Y} \longrightarrow Z$. Here $Z \cong P^1$, because X is rationally connected (see [14]).

(2) $dimZ = 2$. In this case general $l \in \{l\}$ is smooth and does not intersect with another general l on Y_1. And it is exactly the general fiber of the RC-fibration.

We will proceed by the following way. First of all we will treat the case (1). Doing this we will require $l \cdot (-K_X)$ to be bounded not by 6 but only by an arbitrary constant depending only on n. After that we will reduce the case (2) to the case (1) but for some new family $\{l'\}$ where $l' \cdot (-K_X)$ will be bounded.

4. The treatment of case (1)

Let S be a general fiber of our RC-fibration. As we already mentioned, the image of RC-fibration is rational. This implies that S are linear equivalent on $\tilde{Y}$ and therefore on X. Notice that it can happen that $\{l\}$ does not connect two general points of S immediately. But it will always be true if we glue two copies of $\{l\}$. (See [14].) Therefore we will assume that $\{l\}$ is a connecting family on S. Evidently, $l^2 \geq 1$ on a smooth surface $\tilde{S} = S_{\tilde{Y}}$. The condition that X is $\mathbb{Q}$-factorial with Picard number 1 implies that $S_X = \alpha H, \alpha > 0$, where $H = (-2nK_X)$. We may assume up to the end of this section that $l \cdot H \leq \rho$, where ρ is some constant depending on n.

Proposition 4.1. *If $h^0(H) > 2(\rho+1)^2$ then $\alpha \leq \frac{1}{2}$.*

Proof. Let S_1 and S_2 be two general surfaces from $\{S_X\}$. Let $l_1, l_2, ...l_{\rho+1} \subset S_1$ and $l_{\rho+2}, l_{\rho+3}, ...l_{2\rho+2} \subset S_2$ be general curves from $\{l\}$. We have that $H \cdot l \leq \rho$, therefore

$$dimH^0(O_X(H)) - dimH^0(J_{l_i} \cdot O_X(H)) \leq dimH^0(O_{l_i}(H)) \leq \rho + 1.$$

(Here J_{l_i} is an ideal sheaf of the curve $l_i \subset X$.)

This implies that

$$codim(\bigcap_{i=1}^{2\rho+2} H^0(J_{l_i} \cdot O_X(H))) \leq \sum_{i=1}^{2\rho+2} (\rho+1) = 2(\rho+1)^2.$$

If $h^0(H) > 2(\rho+1)^2$ then there exists a divisor $H^* \in |H|$, such that all $l_i \subset H^*$. Suppose π_1 is a composition of the birational morphism $\tilde{S_1} \longrightarrow S_1$ and the embedding $S_1 \longrightarrow X$. If H^* does not contain S_1 then $(\pi_Y^{\tilde{Y}})^*(H^*)$ does not contain $\tilde{S}_1$ but at the same time contains preimages of l_i. On $\tilde{S}_1$ we have $(\pi_1^* H^*) \cdot l \geq \sum l_i \cdot l \geq \rho + 1$. It contradicts to the fact that $\pi_1^*(H^*) \cdot l = H \cdot l \leq \rho$. Therefore H^* contains S_1 and, by the same arguments, S_2. This implies that $\alpha \leq \frac{1}{2}$.

Thus, we may assume below that $\alpha \leq \frac{1}{2}$.

Proposition 4.2. *For an arbitrary $S \in \{S\}$ on X, an arbitrary positive integer k*

$h^i(X, J_S \cdot O_X(kH)) = 0$ for every $i > 0$.

Proof. S is a prime divisor, therefore $J_S = O_X(-S)$, where $O_X(-S)$ is a divisorial sheaf, associated with Weil divisor $(-S)$. After that, $O_X(kH)$ is an invertible sheaf, therefore $J_S \cdot O_X(kH) = O_X(kH - S)$. Now one can apply Kawamata-Vieweg vanishing theorem (see [11], $1-2-5$) because $kH - S - K_X = (k - \alpha + \frac{1}{2n})H$ is ample for $k \geq 1$.

Proposition 4.3. *For all $k > 0$, $l > 0$, $h^i(S, O_S(kH)) = O$.*

Proof. It follows from the exact sequence
$0 \longrightarrow J_S \cdot O_X(kH) \longrightarrow O_X(kH) \longrightarrow O_S(kH) \longrightarrow 0$, vanishing theorem and proposition 4.2.

Proposition 4.4. *All general surfaces S_X for given n and $\rho = c(n)$ lie in finite number of families.*

Proof. By the definition, general S_X are reduced. Thus by the result of Kollár ([12]) and proposition 4.3 it is enough to prove the boundedness of coefficients of Hilbert polynomial $P(k) = \chi(O_S(kH)) = h^0(O_S(kH)), k \geq 1$. For this purpose we will prove that there exists some constant $c_1(n, \rho)$ such that for all $k \geq 1$ it is true that $h^0(O_S(kH)) \leq k^2 \cdot c_1(n, \rho)$. It implies the boundedness of coefficients by the following arguments. Suppose $P(k) = a_2k^2 + a_1k + a_0$. Evidently, $0 \leq a_2 \leq c_1$. Therefore $|a_1| = |P(2) - P(1) - 3a_2| \leq 4c_1$. After that, $|a_0| = |P(1) - a_2 - a_1| \leq 5c_1$.
In order to prove that $h^0(O_S(kH)) \leq k^2 \cdot c_1(n, \rho)$ consider the following construction. By applying several times gluing lemma to a free family $\{l\}$ on $\tilde{S}$ ([14]) we obtain families $\{l_k\}$ such that $l_k = k \cdot l$ as divisors on $\tilde{S}$ and therefore on S. (Here "=" means algebraic equivalence.) Notice that the natural map $\mu_k : H^0(S, O_S(kH)) \longrightarrow H^0(l_{k\rho+1}, O_{l_{k\rho+1}}(kH))$ is injective. Otherwise there should have been some $D \in |kH|$ containing $l_{k\rho+1}$ but not containing S. As in the proof of proposition 4.1 we obtain a contradiction by intersecting with general l. The fact that μ_k is injective implies that $h^0(O_S(kH)) \leq h^0(l_{k\rho+1}, O_{l_{k\rho+1}}(kH)) \leq l_{k\rho+1}(kH) + 1 = (k\rho + 1)k\rho + 1$, that is what we need.

Proposition 4.5. *In the condition of the above proposition there is a constant $c_2(n, \rho)$, such that on every general S_X EVERY two points can be joined by some irreducible curve γ, such that $\gamma \cdot (-K_X) \leq c_2$.*

Proof. It is a straightforward consequence of boundedness of S_X with $H|_{S_X}$. Indeed, it is true for a general element of every one of families in proposition 4.4 and the Noetherian induction on the base completes the proof.

Remark 4.1. *Of course, two GENERAL points of S_X are already connected by l, but the above proposition gives much more.*

Now we can complete the treatment of the case (1). By the definition of Fano log-threefold, we have $\rho(X) = 1$ therefore two general S_X intersect with each other. Moreover, they intersect along some curve C because X is $\mathbb{Q}$-factorial. We know that $\{S_X\}$ is a linear system, therefore all of them contain C. It may happen that C lies in $Sing(X)$, but the multiplicity of X in a general point $x_0 \in C$ is bounded by $2n$, because the index of X is bounded by n. (By the canonical cover trick it is a factor of CDV singularity that is analytically isomorphic to $(DV - point) \times (disk)$.) Therefore we can apply lemma 2.2 to $X, (-K_X), x_0$ to obtain a bound on $(-K_X)^3$.

5. Accurate resolution

In this section we will prove the following lemma.

Lemma 5.1. *(accurate resolution) Suppose X is a $\mathbb{Q}$-factorial three-dimensional variety, $E \subset X$ is a prime Weil divisor, $\{L\}$ is a covering family of curves on E. Suppose further that there exists a covering family $\{l\}$ on X, such that $l \cdot E \geq 1$ and a linear system $|H|$ on X, such that the following inequalities hold true. (c_i are some nonnegative constants.)*

1. $H \cdot l \leq c_1$
2. $H \cdot L \leq c_2$
3. $K_X \cdot L \leq c_3$
4. $-E \cdot L \leq c_4$.

Then $h^0(H) > 1 + (c_1 + 1)(c_2 + c_1c_4 + 1)$ implies that there exists a resolution $Y \longrightarrow X$, such that $\{L\}$ has no base points on E_Y and $K_Y \cdot L \leq c_3 + 2(c_2 + c_1c_4)$.

Remark 5.1. *In some sense this lemma is a very weak substitute for the following conjecture for which I have a lot of evidence.*

Accurate Resolution Conjecture. *For an arbitrary $\mathbb{Q}$-Gorenstein threefold X there exists a resolution of singularities $\pi : Y \longrightarrow X$, such that for EVERY prime Weil $\mathbb{Q}$-Cartier divisor D on X containing a curve L_X not lying in Sing(X) the following inequality holds true.*

$$(K_Y + D_Y) \cdot L_Y \leq (K_X + D_X) \cdot L_X.$$

First of all let me introduce some convenient notation. Let $\{D\}$ be a linear system of Weil divisors. We will denote by $H^0(\{D\})$ the corresponding vector subspace in $H^0(O_X(D))$, where $O_X(D)$ is a divisorial sheaf, associated with D. Reversely, for a linear subspace $V \subset H^0(O_X(D))$ let $|V|$ be the corresponding linear system. A divisor that corresponds to $s \in H^0(O_X(D))$ will be denoted by (s). A section that determines a divisor D will be called an "equation" of D. Of course, it is defined up to a multiplicative constant. By definition $h^0(\{D\}) = dim H^0(\{D\}) = dim\{D\} + 1$.

For the purpose of convenience we introduce the concept of the L-base of a linear system in the following way. Suppose $\{D\}$ is a linear system of Weil divisors, $\{L\}$ is a family of curves parameterized by the base S. For every nonempty Zariski open subset $U \subset S$ let $V(U, \{D\})$ be the linear subspace in $H^0(\{D\})$, spanned by s, such that (s) contains L_u for some $u \in U$. Evidently, $V(U' \bigcap U'', \{D\}) \subset V(U', \{D\}) \bigcap V(U'', \{D\})$ and $H^0(\{D\})$ is finite-dimensional. Therefore there exists the minimal $V(U^*, \{D\})$, such that $V(U^*, \{D\}) \subset V(U, \{D\})$ for every $U \subset S$. Then $|V(U^*, \{D\})|$ will be called L-base of $\{D\}$ and denoted by $\{D\}^L$.

Proposition 5.1. $h^0(\{D\}^L) \geq h^0(\{D\}) - L \cdot D - 1$.

Proof. Suppose $\{D\}^L = |V(U^*, \{D\})|$, $u \in U^*$. We can also assume that L_u is not contained in $Sing(X)$. Choose on L_u points $x_1, x_2, \ldots, x_d$, $L \cdot D < d \leq L \cdot D + 1$ lying in the nonsingular part of X. The condition of vanishing at $x_1, x_2, \ldots, x_d$ determines a subspace in $H^0(\{D\})$ of codimension no greater than d and $d \leq L \cdot D + 1$. Now we just notice that for every s from this subspace (s) contains L_u, because otherwise we would have a contradiction by intersecting it with L_u.

Define a new linear system $\{H_*\}$ by the following procedure. Denote $|H|$ by $\{H_0\}$ and for every nonnegative integer i let $\{H_{i+1}\}$ be a movable part of $\{H_i\}^L$. Evidently, $\{H_i\}$ will eventually stabilize. This stabilized $\{H_i\}$ will be our $\{H_*\}$. It is evident that $\{H_*\}$ is movable and $\{H_*\} = \{H_*\}^L$. (Here we set as definition that trivial linear systems $\emptyset$ and $|O_X|$ are movable.)

Proposition 5.2. *If $h^0(H) > 1 + (c_1+1)(c_2 + c_1c_4 + 1)$ then $\{H_*\}$ is not trivial.*

Proof. First of all, let $\{H_i\}^L = a_iE + D_i + \{H_{i+1}\}$, where $a_i \geq 0$, D_i does not contain E. Notice that if $a_i = 0$ then $\{H_{i+1}\}^L = \{H_{i+1}\}$ and the procedure stabilizes. On the other hand, $\sum a_i \leq c_1$ because $E \cdot L \geq 1$ and $H \cdot l \leq c_1$. Therefore $\{H_*\} = \{H_{[c_1]+1}\}$. It is easy to see that for all i $H_i \cdot L \leq H \cdot L + (\sum_{j=0}^{i} a_j)c_4 \leq c_2 + c_1c_4$. Therefore by proposition 5.1 we have that $h^0(\{H_*\}) \geq h^0(H) - (c_1 + 1)(c_2 + c_1c_4 + 1) > 1$. This implies $\{H_*\}$ is not trivial.

We also have from the above proof that $H_* \cdot L \leq c_2 + c_1c_4$. Now we are going to use the variant of Minimal Model Program invented by Alexeev ([1], 1.8) which I will call Alexeev Minimal Model Program. Let us apply it to $K_X + 2\{H_*\}$. Namely, let $\pi : Y_1 \longrightarrow X$ be a terminal modification of $K_X + 2\{H_*\}$ in sense of Alexeev.

Proposition 5.3. *Under the above notation the following is true.*

1. *Y_1 is $\mathbb{Q}$-factorial and has at worst terminal singularities.*
2. *$\{\pi' H_*\}$ is free. Here $\{\pi' H_*\}$ is the inverse image of the linear system $\{H\}$ in sense of Alexeev, that is a general element of $\{\pi' H_*\}$ is $\pi' H_*$ for a general $H_* \in \{H_*\}$.*
3. *$K_{Y_1} \cdot L \leq c_3 + 2(c_2 + c_1c_4)$.*

Proof. Parts (1) and (2) can be easily proved in the same way as Lemma in [1], 1.22. (This is an application of the Kawamata's result about the minimal discrepancies of 3−dimensional terminal singularities, see [10].) Part (3) is a corollary of the following chain of inequalities.

$K_{Y_1} \cdot L \leq (K_{Y_1} + 2(\pi' H_*)) \cdot L \leq (K_X + 2H_*) \cdot L \leq c_3 + 2(c_2 + c_1c_4)$

Here the middle inequality is due to the following argument. By the definition of terminal modification $K_{Y_1} + 2(\pi' H_*)$ is $\pi - nef$ and therefore in adjunction formula $K_{Y_1} + 2(\pi' H_*) = \pi^*(K_X + 2H_*) + \sum a_iD_i$, where D_i are exceptional divisors, all $a_i \leq 0$.

For the rest of the section we will use the following notation. Suppose D_i, $i = 1, ..., k$ are exceptional divisors of morphism π. For an arbitrary Weil divisor F on X we will say that discrepancy of F is an ordered set $\{discr_{D_i}(F)\}$ of discrepancies of F in D_i, that is numbers $discr_{D_i}(F)$ from the formula $\pi^*F = \pi'(F) + \sum discr_{D_i}(F)D_i$. In this notation we have the following lemma.

Lemma 5.2. *Suppose $F = (s), s \in H^0(O_X(F))$. Suppose $s = \sum \alpha_j s_j$, where $(s_j) = F_j$. Then for all D_i $discr_{D_i}(F) \geq \min_j discr_{D_i}(F_j)$ and for a general $\{\alpha_j\}$ for given $\{s_j\}$ this inequality becomes an equality.*

Proof. Suppose rF is a Cartier divisor. In a neighborhood of the generic point $\pi(D_i)$ the sheaf $O_X(rF)$ can be trivialized. With respect to this trivialization the local equation f of the divisor rF is, by the Newton binomial formula, a linear combination of local equations $f_{(\gamma)}$ of divisors $\sum \gamma_j F_j$, where $\sum \gamma_j = r$, $\gamma_j \in \mathbf{Z}_{\geq 0}$. By the definition, $discr_{D_i}(F) = \frac{1}{r} discr_{D_i}(rF)$ and $discr_{D_i}(rF)$ is just an image of f by a valuation on the function field $\mathbf{C}(X)$ of the variety X corresponding to D_i. Therefore for an arbitrary $\{\alpha_j\}$ $discr_{D_i}(F) \geq \frac{1}{r} \min_j discr_{D_i}(\sum \gamma_j F_j) \geq \min_j discr_{D_i}(F_j)$ and for general $\{\alpha_j\}$ it becomes an equality.

Suppose now that $P_1 \subset \{H_*\}$ is a set of all divisors H_* containing some $L \in \{L\}$. Suppose a general element of P_1 has discrepancy $\{d_i\}$. Denote the set of all divisors from P_1 with such discrepancy by P.

Proposition 5.4. *"Equations" of H_*, $H_* \in P$, span $H^0(\{H_*\})$.*

Proof. For a general $L \in \{L\}$ divisors $H_* \in P$, containing L constitute a nonempty Zariski open subset in the linear system of divisors from $\{H_*\}$ containing L. Therefore their "equations" span the corresponding subspace in $H^0(\{H_*\})$. By definition $\{H_*\} = \{H_*\}^L$, so we are done.

Proposition 5.5. *$\{L\}$ has no base points on E_{Y_1}.*

Proof. Proposition 5.4 and lemma 5.2 applied together imply that discrepancy of any general element of linear system $\{H_*\}$ equals $\{d_i\}$. Therefore for every $H_* \in P$ $\pi' H_* \in \{\pi' H_*\}$. Moreover, the linear equivalence between divisors $\pi' H_*$ is given by the same functions from $\mathbf{C}(Y_1) = \mathbf{C}(X)$ as between corresponding divisors H_*. Therefore the proposition 5.4 implies that "equations" of $\pi' H_*$, where $H_* \in P$, span $H^0(\{\pi' H_*\})$.

Suppose all L on Y_1 pass through some point y. Then all $\pi' H_*$, where $H_* \in P$, contain y. But it is in contradiction with proposition 5.3, (2), so proposition 5.5 is proven.

To complete the proof of the whole Accurate Resolution Lemma it is enough to choose an arbitrary resolution of singularities $Y \longrightarrow Y_1$. Then $Y \longrightarrow X$ will satisfy all the requirements of accurate resolution.

6. Treatment of case (2)

Lemma 6.1. *(adjunction) Suppose X is a 3−dimensional Cohen-Macaulay variety and S is a prime Weil divisor on it, such that $(K_X + S)$ is $\mathbb{Q}$-Cartier. Suppose $\{L\}$ is a covering family of curves on S, $\hat{S}$ is the minimal resolution of the normalization of S. Then $K_{\hat{S}} \cdot L \leq (K_X + S) \cdot L$.*

Proof. Denote by $\tilde{S}$ the normalization of S. Denote by π the natural morphism $\tilde{S} \longrightarrow X$. Then it follows from the Subadjunction Lemma ([11], 5-1-9) that $K_{\tilde{S}} = \pi^*(K_X + S) - D$, where D is an effective $\mathbb{Q}-$Cartier divisor. The rest is trivial.

Now we are in situation and notation of case (2). (See section 3.)

Proposition 6.1. *On Y_1 there exists a divisor E which is exceptional with respect to morphism $\pi_X^{Y_1} : Y_1 \longrightarrow X$ such that $E \cdot l \geq 1$.*

Proof. Suppose C is some general enough curve on the image Z of RC-fibration ϕ. Suppose $D \subset X$ is the image by $\pi_X^{Y_1}$ of the surface $[\phi^{-1}(C)]$. (Here parenthesis means Zariski closure.) The general l_{Y_1} does not intersect $\phi^{-1}(C)$ and, therefore, $[\phi^{-1}(C)]$. (Y_1 is smooth therefore $\{l\}$ is free, see [16].) So, if l_{Y_1} does not intersect with exceptional divisors of $\pi_X^{Y_1}$ then $l_X \cdot D = 0$, that is impossible because X is $\mathbb{Q}$-factorial and $\rho(X) = 1$. Q.E.D.

Notice that if $E \cdot l \geq 1$ then general l_{Y_1} intersects with E in general points because $\{l_{Y_1}\}$ is free. Two cases are possible.

(A) There exists such $E \subset Y_1$ that is exceptional with respect to the morphism $\pi_Y^{Y_1}$.

(B) The family $\{l_Y\}$ is free. Then there exists $E \subset Y$ that is exceptional with respect to π_X^Y.

The proof is generally the same in both cases but some technical details are different. We begin with the case (A). By the relative version of the usual Minimal Model Program the morphism $\pi_Y^{Y_1}$ can be decomposed into extremal contractions and flips, relative over Y. Suppose $\pi_{Y_2}^{Y_3}$ is the first that contracts some divisor E_{Y_3}, for which $l \cdot E_{Y_3} \geq 1$. Suppose $\hat{E}_{Y_3}$ is the minimal resolution of the normalization of E_{Y_3}.

Proposition 6.2. (Case (A)) *There exists a covering family $\{L\}$ of rational curves on E_{Y_3}, such that the following conditions hold true.*

1. $L \cdot K_{Y_3} < 0$
2. $-L \cdot E_{Y_3} < 3$
3. *L does not admit a nontrivial 2-point deformation on $\hat{E}_{Y_3}$, that is a deformation with two fixed points, whose image is not in L.*

Proof. *Suppose $\pi_{Y_2}^{Y_3}(E_{Y_3})$ is a curve. Then we can choose $\{L\}$ to be the fibers of $\pi_{Y_2}^{Y_3}|_{E_{Y_3}}$. Then (1) is true by the definition of an extremal contraction. Suppose $\tilde{E}_{Y_3}$ is a normalization of E_{Y_3}. Then $\{L\}$ does not have base points on $\tilde{E}_{Y_3}$ and therefore L does not pass through its singularieties. This easily implies (3). The condition (2) follows from the fact that (by lemma 6.1)*

$$(K_{Y_3} + E_{Y_3}) \cdot L \geq K_{\hat{E}_{Y_3}} \cdot L = -2 > -3.$$

Suppose now that $\pi_{Y_2}^{Y_3}(E_{Y_3})$ is a point. Consider a minimal model F of $\hat{E}_{Y_3}$. The surface $\hat{E}_{Y_3}$ is birationally ruled or rational therefore we have two possibilities for F:

1. $F \cong P^2$
2. *F is ruled, there is a morphism $\theta : F \longrightarrow C$*

We let $\{L\}$ be the family of lines on P^2 in the first case and the family of fibers of θ in the second one. It evidently satisfies the condition (3). The condition (1) holds for arbitrary curve on E_{Y_3}. The condition (2) again follows from the fact that

$$(K_{Y_3} + E_{Y_3}) \cdot L \geq K_{\hat{E}_{Y_3}} \cdot L \geq -3.$$

The proposition is proven.

Now we can apply the Accurate Resolution Lemma (lemma 5.1). Here X means Y_3, H means $(\pi_X^{Y_3})^*(-2nK_X)$ and constants will be as follows.

$c_1 = 12n$, $c_2 = 0$, $c_3 = 0$, $c_4 = 3$.

We see that if $h^0(-2nK_X)$ is big enough there exists a resolution $Y_4 \longrightarrow Y_3$ such that $K_{Y_4} \cdot L \leq 2(3 \cdot 12n) = 72n$ and $\{L\}$ has no base points on E_{Y_4}.

Proposition 6.3. *L does not admit a nontrivial 2-point deformation on Y_4.*

Proof. If such deformation existed it would be a deformation on E_{Y_4} by rigidity lemma. (About this lemma see [6], section 1. I must only notice that it is not stated there correctly, one should add a condition of flatness of morphism f. It was noticed by several people, my attention was brought to it by Iskovskikh.) The system $\{L\}$

has no base points on E_{Y_4} therefore L does not pass through the singularities of normalization $\tilde{E_{Y_4}}$ of the surface E_{Y_4}. Any resolution of singularities $\hat{E_{Y_4}}$ is naturally mapped to $\hat{E_{Y_3}}$ therefore 2-point deformation of L on E_{Y_4} gives deformation on $\tilde{E_{Y_4}}$ and then on $\hat{E_{Y_4}}$, and then on $\hat{E_{Y_3}}$. The last is impossible by the choice of L, Q.E.D.

Now we can apply to $\{L\}$ and $\{l\}$ the gluing lemma on Y_4 (see [13]) to obtain a new covering family of rational curves $\{l'\}$. But now the image of RC-fibration corresponding to $\{l'\}$ is of dimension 1 or 0. And

$$l' \cdot (-K_X) \leq (1 + dimY_4 + L \cdot K_{Y_4})(l \cdot (-K_X)) \leq 6(4 + 72n).$$

So we managed to reduce the case $(2A)$ to cases (1) and (0), as it was promised at the end of section 3.

Now we consider the case (B). Similarly to the case (A), we have the following statement.

Proposition 6.4. (Case (B)) *There exists a covering family $\{L\}$ of rational curves on E_Y, such that the following conditions hold true.*

1. $L \cdot K_Y < 0$.
2. $-L \cdot E_Y < 3$.
3. *L does not admit a 2-point nontrivial deformation on $\hat{E}_{Y_3}$.*
4. *$\pi_X^Y(L)$ is a point.*

Proof. If $\pi_X^Y(E_Y)$ is a curve let $\{L\}$ be the family of fibers of $\pi_X^Y|_{E_Y}$. If $\pi_X^Y(E_Y)$ is a point then let it come from the minimal model of $\hat{E}_Y$ as in the proof of proposition 6.2. As in the case (A), $K_{\hat{E}_Y} \cdot L$ is -2 or -3. Conditions (3) and (4) are evidently satisfied, we only need to prove (1) and (2). In order to do this, consider the adjunction formula for π_X^Y, multiplied by L :

$$K_Y \cdot L = \sum_{E_i \neq E_Y} a_i \cdot E_i L + aE_Y \cdot L \ (*)$$

Here a_i and a are discrepancies, they are of form $(-\frac{m}{n})$, $m \in \{0, 1, ..., n-1\}$, where n is an index of X. (Discrepancies are nonpositive because Y is a terminal modification of X.) We have the following chain of inequalities.

$$-3 \leq K_{\hat{E}_Y} \cdot L \leq (1 + a)E_Y \cdot L + \sum_{E_i \neq E_Y} a_i E_i \cdot L \leq (1 + a)E_Y \cdot L$$

Here the middle inequality follows from lemma 6.1 and formula $(*)$, and the right from nonpositivity of a_i. Therefore $1 + a \geq \frac{1}{n}$ implies that either $-E_Y \cdot L \leq 0$ or $-E_Y \cdot L \leq 3n$. Therefore $-E_Y \cdot L \leq 3n$. Now the condition (1) follows from the following chain of inequalities.

$$K_Y \cdot L = \sum_{E_i \neq E_Y} a_i E_i \cdot L + aE_Y \cdot L \leq aE_Y \cdot L \leq 3n$$

Here the right inequality holds because of the following argument. We know that $-1 < a \leq 0$ therefore $E_Y L \geq 0$ implies $aE_Y L \leq 0$ and $E_Y L < 0$ implies $aE_Y L \leq -E_Y L$. Q.E.D.

Again, as in case (A), we apply the Accurate Resolution Lemma (lemma 5.1). The only difference is that now we have Y instead of Y_3 and constants are as follows.

$c_1 = 12n$, $c_2 = 0$, $c_3 = 3n$, $c_4 = 3n$.

Again if $h^0(-2nK_X)$ is big enough there exists an accurate resolution Y_4. We have again that L does not admit nontrivial 2-point deformation on Y_4. (Arguments from the proof of proposition 6.3 work without any problems because of condition (4) of proposition 6.4.) So we can apply gluing lemma from [13]. The bound on $l' \cdot (-K_X)$ will be the following.

$l' \cdot (-K_X) \leq (4 + L \cdot K_{Y_4})(l \cdot (-K_X)) \leq (4 + 3n + 2(12n \cdot 3n)) \cdot 6n = 6n(4 + 3n + 72n^2)$.

So we completed the treatment of case $(2B)$. Our Main Theorem is finally proven.

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