CONVOLUTION STRUCTURES AND ARITHMETIC COHOMOLOGY

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Revised version

1. Introduction

In the beginning of 1998 Gerard van der Geer and René Schoof posted a beautiful preprint (cf. [2]). Among other things in this preprint they defined exactly $h^0(L)$ for Arakelov line bundles L on an "arithmetic curve", i.e. a number field. The main advantage of their definition was that they got an **exact** analog of the Riemann-Roch formula $h^0(L) - h^0(K-L) = \deg L + 1 - g$. Before that $h^0(L)$ was defined as an integer and the Riemann-Roch formula above was only true approximately (cf. [6]). However van der Geer and Schoof gave no interpretation for $h^1(L)$ except via duality. They indicated this as one of the missing blocks of their theory. In this paper we go even further to develop the interpretations for $H^0(L)$ and $H^1(L)$ as well as their dimensions. The main features of our theory are the following.

1) H^1 is defined by a procedure very similar to Ĉech cohomology.

2) We get separately Serre's duality and Riemann-Roch formula without duality.

3) We get the duality of $H^0(L)$ and $H^1(K-L)$ as Pontryagin duality of convolution structures.

4) The Riemann-Roch formula of van der Geer and Schoof follows automatically from our construction by an appropriate dimension function.

The paper is organized as follows. In section 2 we define our basic objects (ghost-spaces) and their dimensions. In section 3 we introduce some short exact sequences of ghost-spaces. In section 4 we develop the duality theory of ghost-spaces. In section 5 we apply the theory to arithmetic and obtain our main results. In section 6 we discuss possible directions in which the theory can grow.

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2. Ghost-spaces and their dimensions

Here we define objects that will play the major role in the rest of this paper. We will call them ghost-spaces. Please see Remark 2.4 for some justification of the term.

Basically, ghost-space is a pair $(G, *)$ where G is a locally compact commutative group and $*$ is some commutative and associative convolution of measures structure on it. By the convolution of measures structure we mean the map from $G \times G$ to the space of bounded measures on G,

$$* : (x, y) \to \delta_x * \delta_y$$

We call a convolution associative if it comes from some convolution algebra of measures which contains the space of bounded measures. The convolution above will always be weakly separately continuous, where weak topology is defined using the functions with compact support.

Remark 2.1. *The reader primarily interested in the arithmetic applications can completely disregard the analytic part of our theory. In fact, our convolution structures will always be given by explicit formulas, and the fact that they extend to some measure algebras will never be used.*

For the purpose of this paper, we only need ghost-spaces of two kinds, cf. Definitions below. In order to extend the theory to higher dimensions one would need to allow more complicated convolution structures. Please refer to the Example in section 6. This more general theory will hopefully unify the two kinds of ghost-spaces we currently have. Unfortunately, it is not fully developed yet.

In what follows, we will use the notion of functions and measures of positive type (positive definite functions and measures). This is a standard and pretty well understood notion in harmonic analysis (cf. [1]). Roughly speaking, it means having nonnegative Fourier transform. But if you are primarily interested in arithmetic applications and are not comfortable with this notion, don't worry about it. In applications it will be automatically satisfied.

Lemma 2.1. *Suppose G is a locally compact abelian group. Suppose $u : G \to R^+$ is a positive symmetric continuous function of positive type on it such that $u(0) = 1$. Consider the convolution of measures $*$*

on G such that

$$\delta_x * \delta_y = \frac{u(x)u(y)}{u(x+y)}\delta_{x+y}$$

Then this convolution is commutative and associative.

Proof. Consider the space of all Radon measures μ with the property that $u\mu$ is bounded Them we can make it a convolution algebra by setting

$$\mu_1 * \mu_2 = \frac{(u\mu_1) \odot (u\mu_2)}{u},$$

where $\odot$ is the standard convolution of measures on G. This convolution $*$ extends the convolution $\delta_x * \delta_y$. It is obviously commutative, and associative. It is also weakly separately continuous, where the weak topology is defined using the continuous functions with compact support. $\square$

Remark 2.2. *Function u being of positive type or positive is not really necessary for the above theorem, being bounded is. Please also cf. Voit ([8]) for a related much more general theory. We just stated the above lemma in the case that we are going to need in this paper.*

Remark 2.3. *In fact, any continuous real-valued function of positive type is symmetric (cf. [1], prop. 3.22). Also, in many other sources, functions of positive type are called positive-definite functions. In the terminology of Folland [1] positive-definite is a bit weaker condition. Though for the continuous functions there is no difference anyway.*

Definition 2.1. *We will call the pair $(G, *)$ as above the ghost-space of the first kind, to be denoted G_u. We also define the dimension of G_u which depends on the choice of a Haar measure m on G. Namely,*

$$\dim_m G_u = \log \int_G u(x)dm(x)$$

When G is discrete, it has a distinguished Haar measure, the counting measure m_c. In this case we will say that the dimension of G_u

$$\dim G_u = \dim_{m_c} G_u.$$

Examples.

1) Suppose G is a locally compact abelian group. Then G_1 is just G itself with the standard convolution of measures. We will therefore identify G_1 with G.

2) Suppose $G = \mathbb{Z}^n$ and Q is a positive-definite quadratic form on it. Then one can check that $u(x) = e^{-Q(x,x)}$ is of positive type. (This follows from the positivity of its Fourier transform (cf. Theorem 5.2

of this paper). So one can define the ghost space G_u. Its dimension, in the above sense, is equal to $\log \sum_{x\in\mathbb{Z}} e^{-Q(x,x)}$. This is exactly the kind of formula van der Geer and Schoof used to define $h^0(D)$, and $u(x)$ is their effectivity function. So the finite-dimensional ghost-space of the first kind G_u is going to be, in our interpretation, $H^0(D)$.

Remark 2.4. *The above example justifies somewhat the word "ghost-space". Indeed, one can think of G_u as a space whose elements do not exactly belong to the real world. So they come with the "effectivity function" that measures how real they are. In the above example the only 100% real element is* 0. *Also, the following theorem shows that effectivity is always at most* 1.

Theorem 2.1. *Suppose G_u is a ghost-space of the first kind. Then for all $x \in G$ $u(x) \leq 1$. Also, those x that $u(x) = 1$ form a closed subgroup H of G. Moreover, $u(x)$ comes from a function on G/H.*

Proof. The first claim is contained in Folland [1], cor. 3.32. To prove the second and third claims we note that by [1], prop. 3.35 the following matrix is positive definite.

$$\begin{bmatrix} 1 & \mathrm{u(x)} & \mathrm{u(x{+}y)} \\ \mathrm{u(x)} & 1 & \mathrm{u(y)} \\ \mathrm{u(x{+}y)} & \mathrm{u(y)} & 1 \end{bmatrix}$$

If $u(x) = 1$, it implies that $(u(x+y) - u(y))^2 \leq 0$, so $u(x+y) = u(y)$. This implies the theorem. $\square$

Now we define the ghost-spaces of the second kind. While the ghost-spaces of the first kind are intuitively the abelian groups with "partially existent" elements, the ghost-spaces of the second kind have different nature. They are the abelian groups with good elements but bad addition. Namely the addition is in general faulty with the error probability being "translation invariant".

Lemma 2.2. *Suppose G is a locally compact abelian group. Suppose μ is a symmetric positive probability measure of positive type on G. Consider the convolution of measures $*$ on G such that*

$$\delta_x * \delta_y = T_{x+y}\mu,$$

where T_{x+y} is the usual shift by $(x+y)$. Then this convolution is commutative and associative.

Proof. We will show that $*$ extends to the space of bounded measures. We will use for that the canonical continuation formula of Pym (cf. [3]). For any two bounded Radon measures ν_1 and ν_2, and a continuous function with compact support f on G, the following formula makes sense.

$$(\nu_1 * \nu_2)(f) = \int\int (T_{x+y}\mu)(f) d\nu_1(x) d\nu_2(y)$$

One can use it to define the Radon measure $\nu_1 * \nu_2$. This obviously generalizes the convolution $*$ from the statement of the lemma. One can easily check that $\nu_1 * \nu_2$ is bounded. Moreover, the convolution of two probability measures is a probability measure, and the convolution is weakly separately continuous. We now need to check that it is associative. If ν_1, ν_2, ν_3 are bounded Radon measures and f is a continuous function with compact support on G then one can check the following.

$$\Big((\nu_1 * \nu_2) * \nu_3\Big)(f) = \int\int\int \Big(T_{x+y+z}(\mu \odot \mu)\Big)(f) d\nu_1(x) d\nu_2(y) d\nu_3(z),$$

where $\odot$ is the standard convolution of measures on G. The associativity follows. □

Definition 2.2. *We will call the pair $(G, *)$ as above the ghost-space of the second kind, to be denoted G^μ.*

We also define the dimension of G^μ in some particular case. Although this is the only case we will need in this paper, a more general definition would be desirable.

Definition 2.3. *Suppose G is compact, and $\mu = u(x) \cdot m$, where $u(x)$ is a continuous function on G and m is the Haar probability measure on it. Then we define dimension of G^μ as*

$$\dim G^\mu = \log u(0)$$

Some justification of the above definition is provided by Lemma 2.3. The real justification, however, is in Proposition 3.1 and Theorem 4.1.

Example. Suppose G is a locally compact abelian group, and $\mu = \delta_0$ is the point measure at 0 on it. Then the convolution on G_μ is just the standard convolution on G.

Because of the above observation, one can choose to consider G both as a ghost-space of the first and of the second kind. In fact, one can see immediately that this is the only case when a convolution structure can be interpreted in these two ways. Since we want the dimension to be determined by the convolution structure itself, and not by its interpretation, we have to check that $\dim G$ does not depend on the above choice. Because of the strict restrictions in the Definition 2.3 the only case we really need to consider is when G is finite. The following lemma does just that.

Lemma 2.3. *Suppose G is a finite abelian group. Then its dimension as a ghost-space of the first or the second kind is equal to* $\log |G|$.

Proof. We will denote by M the counting measure on G.

1) As a ghost-space of the first kind $G = G_1$. So $\dim G = \dim_M G = \log |G|$. □

2) As a ghost-space of the second kind $G = G^{\delta_0}$. If m is the probability Haar measure on G, then $m = \frac{1}{|G|} M$. So $\delta_0 = h \cdot m$, where $h(0) = |G|$, $h(x) = 0$ for $x \neq 0$. Therefore $\dim G = \dim G^{\delta_0} = \log h(0) = \log |G|$. □

3. Short exact sequences of ghost-spaces

In this section we will define two kinds of short exact sequences of ghost-spaces. We will check that the dimension is additive, whenever defined. We must note that this is probably just a little piece of the more general theory which is yet to be developed.

Definition 3.1. *Suppose G_u is a ghost-space of the first kind. Then we say that G_u is a subspace of G. If $\dim G_u < \infty$ we also say that the quotient G/G_u is the ghost-space of the second kind G^μ, where μ is the probability measure on G proportional to $u(x) \cdot m$. Here m is some (any) Haar measure on G.*

Remark 3.1. *The above definition is valid because $u(x) \cdot m$ is of positive type. We should also note that it is rather reasonable. Basically we just define the convolution on the quotient space by an averaging procedure using the measure $u(x) \cdot m$ on a "subspace" G_u. This is very similar to taking usual quotient of groups, though formally not a generalization of it.*

Proposition 3.1. *The dimension is additive in the above short exact sequence, provided we use the same Haar measure for G and G_u to define it. That is, whenever defined,*

$$\dim_m G = \dim_m G_u + \dim G^\mu.$$

Proof. Because of the Definition 2.3 we only need to consider the case when G is compact. Since changing the Haar measure m has no effect on the validity of the above identity, we can choose m to be the probability measure. If $\dim_m G_u = \log A$ then $\mu = \frac{1}{A} \cdot u \cdot m$. Therefore $\dim_m G = 0$, $\dim_m G_u = \log A$, and $\dim G^\mu = \log(\frac{u(0)}{A}) = -\log A$. The last identity is because $u(0) = 1$ by the definition. □

Now we define another kind of short exact sequences. This time all objects are ghost-spaces of the first kind.

Definition 3.2. *Suppose G is a locally compact abelian group and H is its closed subgroup. Suppose $u : G \to R^+$ is a symmetric continuous function of positive type on G such that $u(0) = 1$. Abusing notation a little bit, we will call the restriction of u to H also u. Then we will say that H_u is a subspace of G_u. If we can define a continuous function of positive type v on G/H as below we will also say that $(G/H)_v$ is the quotient G_u/H_u.*

$$v(xH) = \frac{\int\limits_{y \in H} u(x+y)dm(y)}{\int\limits_{y \in H} u(y)dm(y)},$$

where m is a Haar measure on H.

Remark 3.2. *In fact, v is probably always of positive type, whenever it is defined and continuous. At least it is true if both* $\dim G$ *and* $\dim H$ *are finite, as the following proposition shows.*

Proposition 3.2. *Suppose u and v are continuous functions defined as in Definition 3.2. Suppose that $\int\limits_G u(x)dm_G(x)$ and $\int\limits_H u(x)dm_H(x)$ are both finite. Then v is of positive type.*

Proof. Since $v \in L^1(G/H)$ it is enough to show (cf. [1], 4.17) that

$$\int\limits_{G/H} \chi(y)v(y)dm_{G/H}(y) \geq 0$$

for any character χ on G/H. By the definition of v it is equivalent to saying that

$$\int\limits_G \chi(x)v(x)dm_G(x) \geq 0$$

for all characters χ on G that come from G/H. This now follows from u being of positive type (cf. [1], 4.23). □

Remark 3.3. *The dimension is obviously additive in the above short exact sequence if one chooses the measure on the quotient space as the quotient of measures on G and H.*

Remark 3.4. *Pretty obviously, $G_1/H_1 = (G/H)_1$ whenever defined (i.e. when H is compact). So our definition really is compatible with the usual group quotients.*

Remark 3.5. *One can also define similarly some short exact sequences of the ghost-spaces of second kind. They will be dual to the above short exact sequences in the sense of the next section.*

4. Duality theory of ghost-spaces

Here we develop the duality theory of ghost-spaces. Basically, the dual of G_u is $\widehat{G}^{\hat{u}}$, where $\widehat{G}$ is the Pontryagin dual of G and $\hat{u}$ is the Fourier transform of u. To be precise, $\hat{u}$ is such measure that

$$u(x) = \int\limits_{y\in\widehat{G}} y(x)d\hat{u}(x).$$

The existence of such measure is the Bochner theorem on G (cf., e.g. Folland [1], prop. 4.18). We could have taken this as a definition, of course. But we already had a lot of ad hoc definitions in the previous two sections. So we claim that this duality really is the Pontryagin duality of convolution structures.

We should mention here that a lot of work has been done by researchers in harmonic analysis to extend Pontryagin duality of locally compact abelian groups to the more general convolution structures. We should mention here for reference the survey of Vainerman [7]. It looks like the particular case we need is new. But it is very similar algebraically to the more general case of commutative signed hypergroups, as introduced by Margit Rösler ([4], [5]). To be precise, for any G_u one can define an involution by sending x to $-x$, and a measure $\omega = \frac{m}{u^2}$, where m is some Haar measure on G. Then the triple $(G, \omega, *)$ satisfies the algebraic part of the axioms of a commutative signed hypergroup.

So we will construct the dual of G_u following the construction of Rösler. We are only interested in the algebraic part of the construction, and our convolutions are given by explicit formulas. So we will basically ignore the analytic part of the theory.

First, let us consider all quasi-characters on G. These are the functions $\varphi : G \to \mathbb{C}$ with the following property.

$$\varphi(x)\cdot\varphi(y) = \int\limits_{G} \varphi(\lambda)(\delta_x * \delta_y)(\lambda)$$

In our case this means that

$$\varphi(x)\cdot\varphi(y) = \varphi(x+y)\frac{u(x)u(y)}{u(x+y)}$$

So $\frac{\varphi(x)}{u(x)}$ is a multiplicative function on G. This implies that $\varphi(x) = \chi(x)u(x)$ for some multiplicative function $\chi : G \to \mathbb{C}$.

Now we should consider only the symmetric quasi-characters, i.e. those φ that $\varphi(-x) = \overline{\varphi(x)}$. One can see from the above description of quasi-characters that these are $\varphi_\chi(x) = \chi(x)u(x)$ for some $\chi : G \to S^1$, i.e. for $\chi \in \widehat{G}$.

So we established the natural set-wise isomorphism of $\widehat{(G_u)}$ and $\widehat{G}$. We can therefore transfer the group structure of $\widehat{G}$ onto $\widehat{(G_u)}$. What we really need to do though is to figure out the convolution structure on $\widehat{(G_u)}$. First we can define the Fourier transform and the inverse Fourier transform as in Rösler [4].

Since $\varphi_\chi(x) = \chi(x)u(x)$, for all $x \in G$, we have that

$$\check{\delta_\chi}(x) = \chi(x)u(x),$$

where δ_χ is a point measure at φ_χ.

The convolution of measures in $\widehat{(G_u)}$ should correspond via the inverse Fourier transform to the multiplication of functions on G_u, i.e. to the usual multiplication of functions on G. The only thing we really need to prove is the following proposition.

Proposition 4.1. *Suppose* $\chi_1, \chi_2 \in \widehat{G}$, $x \in G$. *Then*

$$(\chi_1(x)u(x)) \cdot (\chi_2(x)u(x)) = \int\limits_{\chi \in \widehat{G}} \chi(x)u(x)d(T_{\chi_1+\chi_2}\hat{u})(\chi)$$

Proof. The above equality is equivalent to the following.

$$u(x) = \int\limits_{\chi \in \widehat{G}} \frac{\chi(x)}{\chi_1(x)\chi_2(x)} d(T_{\chi_1+\chi_2}\hat{u})(\chi)$$

The right hand side can be rewritten as

$$\int\limits_{\chi \in \widehat{G}} (\chi - \chi_1 - \chi_2)(x)d(T_{\chi_1+\chi_2}\hat{u})(\chi)$$

Using the substitution $\lambda = \chi - \chi_1 - \chi_2$, it is equal to

$$\int\limits_{\lambda \in \widehat{G}} \lambda(x)d\hat{u}(\lambda)$$

Then the desired equality is just the definition of $\hat{u}$. □

One can also check that the natural involution of quasi-characters $\varphi \mapsto \bar{\varphi}$ corresponds to $\chi \mapsto -\chi$. To complete the picture we need to show that $\widehat{\widehat{(G_u)}}$ is naturally isomorphic to G_u. This means that all the symmetric quasi-characters of the convolution structure $\widehat{G}^{\hat{u}}$ are of the form $\chi(x)u(x)$ for some $x \in G$. The following proposition does just that.

Proposition 4.2. *Suppose $f : G \to \mathbb{C}$ is a symmetric quasi-character on $\widehat{G}^{\hat{u}}$. Then $f(x) = \chi(x)u(x)$ for some $x \in G$.*

Proof. Being a quasi-character here means that for all $\chi_1, \chi_2 \in \widehat{G}$

$$f(\chi_1) \cdot f(\chi_2) = T_{\chi_1+\chi_2}\hat{u}(f).$$

Therefore

$$f(\chi_1) \cdot f(\chi_2) = f(0) \cdot f(\chi_1 + \chi_2).$$

This implies that $f(\chi) = v(\chi) \cdot f(0)$, where v is a character on $\widehat{G}$.

Also, since f is symmetric, $f(0) = \bar{f}(0)$, so $f(0) \in \mathbb{R}$. As a result, the condition $f(-\chi) = \bar{f}(\chi)$ implies that $v(-\chi) = \bar{v}(\chi)$ so v takes values in the unit circle S^1. By the Pontryagin duality theorem, $v(\chi) = \chi(x)$ for some $x \in G$.

Finally, $f(0) \cdot f(0) = \hat{u}(v \cdot f(0)$. So $f(0) = \hat{u}(v)$. By the definition of $\hat{u}$, $f(0) = u(x)$, the proposition is proven. □

Remark 4.1. *If we take duals in a short exact sequence of Definition 3.2 we get again a short exact sequence, going in the opposite direction. So the situation is completely parallel to the case of usual locally compact abelian groups.*

Now let's discuss what happens with the dimension when the dual is taken. First of all, $\dim \widehat{G}^{\hat{u}}$ only makes sense if $\widehat{G}$ is compact, and $\hat{u}$ is absolutely continuous with respect to a Haar measure. This means that G is discrete. Then we have the following theorem.

Theorem 4.1. *Suppose G is discrete, G_u is a finite-dimensional ghost-space of the first kind. Then*

$$\dim G_u = \dim \widehat{G_u}$$

Proof. Consider the counting measure m on G. Its dual measure $\hat{m}$ is a probability Haar measure on $\widehat{G}$ (cf., e.g. Folland [1], Prop. 4.24). Then $\hat{u} = f(\chi) \cdot \hat{m}$ where f is the Fourier transform of u relative to the above measures (cf. Folland, [1], prop. 4.21). By definition,

$$\dim \widehat{G}^{\hat{u}} = \log f(0) = \dim G_u$$

□

Remark 4.2. *Even though it might be possible to extend the definition of the dimension of the ghost-spaces of the second kind, the above theorem is not likely to have any generalizations. The following example highlights the major obstacle.*

Example. Suppose $u = e^{-\pi x^2}$ is a function on $\mathbb{R}$, and m is the standard measure on $\mathbb{R}$. Then $\mathbb{R}_u$ is the ghost-space of the first kind and $\mathbb{R}^{um}$ is the ghost-space of the second kind. We have the following short exact sequence of ghost-spaces.

$$0 \to \mathbb{R}_u \to \mathbb{R} \to \mathbb{R}^{um} \to 0$$

We have that $\dim \mathbb{R} = \infty$. For any measure M $\dim_M \mathbb{R}_u$ is finite (equal to zero if $M = m$). By the nature of dimension, we expect that $\dim \mathbb{R}^{um} = \infty$. On the other hand, one can check that $\widehat{\mathbb{R}_u} = \mathbb{R}^{um}$. So we have a duality between a finite-dimensional ghost-space $\mathbb{R}_u$ and an infinite-dimensional ghost-space $\mathbb{R}^{um}$.

5. Arithmetic cohomology via ghost-spaces

First of all, let us fix the same notations as in [2], section 3. For the convenience of a reader we reproduce most of them below.

Our main object is an “arithmetic curve”, i.e. a number field F. An Arakelov divisor D on it is a formal sum $\sum\limits_P x_P P + \sum\limits_\sigma x_\sigma \sigma$, where P runs over the maximal prime ideals of the ring of integers O_F and σ runs over the infinite, or archimedean places of the number field F. The coefficients x_P are in $\mathbb{Z}$ while the coefficients x_σ are in $\mathbb{R}$. The degree $\deg(D) = \sum\limits_P \log(N(P))x_P + \sum\limits_\sigma x_\sigma$.

An Arakelov divisor D is determined by the associated fractional ideal $I = \prod P^{-x_P}$ and by $r_1 + r_2$ coefficients $x_\sigma \in \mathbb{R}$. We can define a hermitian metric on I, and on $I \otimes \mathbb{R} = F \otimes \mathbb{R}$ as in [2]. That is, for $z = (z_\sigma)$

$$||(z_\sigma)||_D^2 = \sum_\sigma |z_\sigma|^2 \cdot ||1||_\sigma^2,$$

where $||1||_\sigma^2 = e^{-2x_\sigma}$ for real σ and $||1||_\sigma^2 = 2e^{-x_\sigma}$ for complex σ. According to van der Geer and Schoof,

$$h^0(D) = \sum_{x \in I} e^{-\pi ||x||_D^2}$$

In accordance with this, we make the following definition.

Definition 5.1. *In the above notations, $H^0(D)$ is the ghost-space of the first kind I_u, where $u(x) = e^{-\pi ||x||_D^2}$.*

Remark 5.1. *To make the above definition valid, we need to check that u is of positive type. This basically follows from the positivity of its Fourier dual, which will be calculated in Theorem 5.2 (cf., e.g. Folland* [1]*). Clearly,* $\dim I_u = h^0(D)$.

Now we are going to define $H^1(D)$. First, let us look at how it can be done in the geometric situation. We have the curve C with the map $\pi : C \to P^1$. Probably the easiest way to calculate $H^1(D)$ in this situation is by $\hat{\mathrm{C}}$ech cohomology. For this we need to cover the curve by affine open sets. One way to do it is to choose two points on P^1, say α and ∞, and consider the open sets $U_0 = \pi^{-1}(P^1 - \infty)$ and $U_1 = \pi^{-1}(P^1 - \alpha)$. Then we have the following four spaces.

$$V_{00} = H^0(D, U_0 \cap U_1)$$

$$V_{10} = H^0(D, U_0)$$

$$V_{01} = H^0(D, U_1)$$

$$V_{11} = H^0(D)$$

Here V_{10} and V_{01} are subspaces of V_{00} and $V_{10} \cap V_{01} = V_{11}$. By the definition of $\hat{\mathrm{C}}$ech cohomology, and since U_0 and U_1 are affine,

$$H^1(D) = V_{00}/(V_{01} + V_{10}) = (V_{00}/V_{10})/(V_{01}/V_{11})$$

Now we try something similar in the arithmetic case. Let us choose $U_0 = \pi^{-1}(\infty)$ and $U_1 = \pi^{-1}(p)$ where p is some prime number. Let us denote by J the localization of I in p. Then the natural analog of V_{11} above is the ghost space I_u for $u(x) = e^{-\pi||x||^2_D}$. The analog of V_{10} is I. The analog of V_{00} is J. The analog of V_{01} would have been J_u, if we managed to define ghost-spaces for the groups like J. Then the $\hat{\mathrm{C}}$ech cohomology of this covering should be

$$(J/I)/(J_u/I_u).$$

Now we have some problems. It looks like the different choices of p should lead to different answers, unless we are willing to complete J to $I \otimes \mathbb{R}$. So this is what we do. Please note that $I \otimes \mathbb{R}$ is a locally compact group, and we have no problems in defining the ghost-space V_{01}. We also have no problems to define other ingredients in the formula using the short exact sequences from section 3. So this is our definition.

Definition 5.2. *For an Arakelov divisor D as above*

$$H^1(D) = ((I \otimes \mathbb{R})/I)/((I \otimes \mathbb{R})_u/I_u)$$

Also, $h^1(D) = \dim H^1(D)$, as the dimension of the ghost-space of the second kind.

We will see that this definition yields a beautiful theory with such attributes of the geometric case as Serre's duality and Riemann-Roch. For this we just need to do some calculations.

Proposition 5.1. *We have that*

$$(I \otimes \mathbb{R})_u / I_u = ((I \otimes \mathbb{R})/I)_v,$$

where for every $\overline{x} \in (I \otimes \mathbb{R})/I$

$$v(\overline{x}) = \frac{\sum\limits_{y \in I} e^{-\pi ||x+y||_D^2}}{\sum\limits_{y \in I} e^{-\pi ||y||_D^2}}$$

Proof. This is just the definition of the quotient from section 3, Definition 3.2. □

Theorem 5.1. *Suppose* Δ *is the absolute value of the discriminant of the number field* F*. Then the first cohomology of an Arakelov divisor* D *is the following ghost-space of the second kind.*

$$H^1(D) = ((I \otimes \mathbb{R})/I)^{\omega},$$

where

$$\omega = \frac{\sqrt{\Delta}}{e^{\deg D}} \cdot \sum_{y \in I} e^{-\pi ||x+y||_D^2} \cdot m,$$

where m *is the Haar probability measure on* $(I \otimes \mathbb{R})/I$*.*

Proof. Obviously ω should be proportional to $\sum\limits_{y \in I} e^{-\pi ||x+y||_D^2} \cdot m$. We just have to scale it to make it a probability measure. We have the following.

$$\int\limits_{\overline{x} \in (I \otimes \mathbb{R})/I} \sum_{y \in I} e^{-\pi ||x+y||_D^2} \cdot dm(\overline{x}) = \int\limits_{x \in I \otimes \mathbb{R}} e^{-\pi ||x||_D^2} dM(x),$$

where M is the measure on $I \otimes \mathbb{R}$ such that I has covolume 1. If M_D is the measure that corresponds to the hermitian metric D, the above integral is equal to

$$\frac{e^{\deg D}}{\sqrt{\Delta}} \cdot \int\limits_{x \in I \otimes \mathbb{R}} e^{-\pi ||x||_D^2} dM_D(x)$$

Now we just need to show that

$$\int\limits_{x \in I \otimes \mathbb{R}} e^{-\pi ||x||_D^2} dM_D(x) = 1.$$

This is a pretty standard calculation. It can be done, e.g. by splitting up into the pieces that correspond to the infinite places of F and using the following two identities.

1) (real factor)

$$\alpha \int\limits_{x\in\mathbb{R}} e^{-\pi\alpha^2x^2}dx = 1$$

2) (complex factor)

$$\alpha \int\limits_{x+iy\in\mathbb{C}} e^{-\pi\alpha^2(x^2+y^2)}dxdy = 1$$

These are very standard identities. The second one follows from the direct calculation in polar coordinates. The first one is essentially the square root of the second one. $\square$

Now we are ready for the Serre's duality theorem. For this we need to recall the definition of the canonical Arakelov divisor K on F. It is defined (cf., e.g. [2]) as having associated fractional ideal ∂^{-1} and zero infinite components. Here ∂ is the different of F.

Theorem 5.2. *(Serre's duality) For any Arakelov divisor D we have the following duality of ghost-spaces.*

$$H^1(D) = H^0\widehat{(K-D)}$$

Proof. First we need to establish duality on the level of underlining locally compact groups. Suppose I is the fractional ideal associated with D. It follows from the definition of K that $(I\otimes\mathbb{R})/I=(F\otimes\mathbb{R})/I$ is dual to $\partial^{-1}I^{-1}$, where ∂ is the different of F. The duality is given by the following pairing ($\overline{x}\in(F\otimes\mathbb{R})/I, y\in\partial^{-1}I^{-1}$).

$$(\overline{x}, y) = e^{2\pi i\mathrm{Tr}(xy)},$$

where $x\in F\otimes\mathbb{R}$ is some representative of $\overline{x}$ and $\mathrm{Tr}(xy)$ is taken in the algebra $F\otimes\mathbb{R}$.

Now in order to prove the theorem we just need to show that for every $y\in\partial^{-1}I^{-1}$

$$e^{-\pi||y||^2_{K-D}} = \int\limits_{\overline{x}\in(I\otimes\mathbb{R})/I} e^{2\pi i\mathrm{Tr}(xy)}d\omega(\overline{x}),$$

where ω is the probability measure from Theorem 5.1. Let's just simplify the right hand side.

$$\int\limits_{\overline{x}\in(I\otimes\mathbb{R})/I} \frac{\sqrt{\Delta}}{e^{\deg D}}\sum_{z\in I} e^{-\pi||x+z||^2_D}e^{2\pi i\mathrm{Tr}(xy)}dm(\overline{x}) = \int\limits_{x\in I\otimes\mathbb{R}} e^{-\pi||x||^2_D}e^{2\pi i\mathrm{Tr}(xy)}dM_D(x)$$

This is a pretty standard integral. For the convenience of a reader, we reproduce the calculations in some details below.

Let us suppose that the infinite part of D is given by the real numbers $(\sigma_1, ...\sigma_{r_1}, \sigma_{r_1+1}, ...\sigma_{r_1+r_2})$. Splitting up the above integral, and $e^{-\pi||y||^2_{K-D}}$ into the product of $r_1 + r_2$ factors corresponding to different σ_i, it is enough to prove the following two lemmas.

Lemma 5.1. *(real factor) For any real σ and y the following identity is true.*

$$\int\limits_{x\in\mathbb{R}} e^{-\pi e^{-2\sigma}x^2+2\pi ixy} \cdot e^{-\sigma} dx = e^{-\pi e^{2\sigma} y^2}$$

Proof. First of all, multiplying x by $e^{-\sigma}$ and y by e^{σ} we can get rid of σ. So we just need to prove that

$$\int\limits_{x\in\mathbb{R}} e^{-\pi x^2+2\pi ixy} \cdot dx = e^{-\pi y^2}.$$

The left hand side can be rewritten as

$$\int\limits_{x\in\mathbb{R}} e^{-\pi(x+iy)^2} \cdot e^{-\pi y^2} dx$$

By contour integration, it is equal to

$$\int\limits_{x\in\mathbb{R}} e^{-\pi x^2} \cdot e^{-\pi y^2} dx = e^{-\pi y^2},$$

the lemma is proven. $\square$

Lemma 5.2. *(complex factor) For any $\sigma \in R$ and $y = y_1 + iy_2 \in \mathbb{C}$ the following identity is true.*

$$\int\limits_{x_1+ix_2\in\mathbb{C}} e^{-2\pi e^{-\sigma}(x_1^2+x_2^2)} e^{4\pi i(x_1y_1-x_2y_2)} \cdot 2e^{-\sigma} dx_1 dx_2 = e^{-\pi\cdot 2e^{\sigma}(y_1^2+y_2^2)}$$

Proof. First of all, multiplying x_1 and x_2 by $e^{-\sigma/2}$, and y_1 and y_2 by $e^{-\sigma/2}$, we can get rid of σ. So we just need to prove that

$$\int\limits_{x_1+ix_2\in\mathbb{C}} e^{-2\pi(x_1^2+x_2^2)} e^{4\pi i(x_1y_1-x_2y_2)} \cdot 2dx_1 dx_2 = e^{-\pi\cdot 2(y_1^2+y_2^2)}$$

The left hand side can be rewritten as

$$\int\limits_{x_1}\int\limits_{x_2} 2e^{-2\pi(x_1-iy_1)^2-2\pi(x_2+iy_2)^2} \cdot e^{-2\pi(y_1^2+y_2^2)} dx_1 dx_2$$

This is equal to $e^{-2\pi(y_1^2+y_2^2)}$ by splitting up the above integral and then proceeding like in the previous lemma. $\square$

So, we established the Serre's duality as the duality of ghost-spaces. The obvious corollary of it, and Theorem 4.1 is the following.

Corollary 5.2. *In the above notations,*
$$h^1(D) = h^0(K - D)$$

Now we obtain the Riemann-Roch formula using the additivity of dimension in the short exact sequences of ghost-spaces from section 3.

Theorem 5.3. *(Riemann-Roch formula)*
$$h^0(D) - h^1(D) = \dim D - \frac{1}{2}\log\Delta$$

Proof. We use the notations of Theorem 5.1. By Proposition 3.1 and Remark 3.3,
$$h^1(D) = \dim H^1(D) = \dim_m(I\otimes\mathbb{R})/I - \dim_m((I\otimes\mathbb{R})/I)_v =$$
$$= -\dim_m((I\otimes\mathbb{R})/I)_v = -(\dim_M(I\otimes\mathbb{R})_u - \dim I_u) =$$
$$= h^0(D) - \dim_M(I\otimes\mathbb{R})_u$$
So we have that
$$h^0(D) - h^1(D) = \dim_M(I\otimes\mathbb{R})_u = \log \int\limits_{x\in I\otimes R} e^{-\pi||x||_D^2} dM(x) = \log\frac{e^{\deg D}}{\sqrt{\Delta}}$$
as in the proof of Theorem 5.1. This proves the theorem. □

So, we recovered the Riemann-Roch theorem of van der Geer and Schoof (first proven by Tate in his thesis). Our approach, of course, gives much more structure. We should also note that instead of using the Poisson summation formula, we basically reproved it along the lines of the usual proof of the Riemann-Roch theorem in the geometric case.

6. Further remarks and open problems

There are many directions in which the theory can be developed further. We list below the most interesting possibilities.

1) We believe that the theory can be extended to the higher-dimensional case, at least to the case of curves over number fields. There we have $H^0(D)$, $H^1(D)$, and $H^2(D)$. We believe that $H^0(D)$ should be a discrete finite-dimensional ghost-space of the first kind. $H^2(D)$ should be a compact ghost-space of the second kind, dual to $H^0(K-D)$. The most troublesome part is $H^1(D)$. If D has geometric degree at least $2g-1$ (for the curves of genus g) then $H^2(D)$ should be trivial, and $H^1(D)$ should be a compact ghost-space of the second kind. If D has negative geometric degree then $H^0(D)$ is trivial, and $H^1(D)$ is a discrete ghost-space of the first kind. However the most interesting case

of $0 \le \deg D \le 2g-2$ is not covered above. In this case we conjecture that there still exists a ghost-space interpretation of $H^1(D)$, which is a locally compact group with the convolution structure that generalizes the structures of the ghost-spaces of the first and second kind as in is the following example.

Example. Suppose G is a locally compact abelian group, u is a symmetric continuous function on it, such that $u(0) = 1$. Suppose also that μ is a symmetric probability measure on G. Then the following convolution structure is commutative and associative.

$$\delta_x * \delta_y = \frac{u(x)u(y)}{u(x+y)} T_{x+y}\mu$$

This higher-dimensional generalization is clearly very important. Ultimately, one would like to translate from geometry such things as Kodaira-Spencer map to get a shot at the *abc*-type results. This will be the subject of the author's future work.

2) It is of some interest to extend the theory from the Arakelov divisors to the more general "coherent ghost-sheaves", whatever this should mean. In particular, there are no serious difficulties in extending the theory to the higher rank locally free sheaves, parallel to the construction of van der Geer and Schoof.

3) As noted in [2], prop. 6, zeta function of F is kind of given by the following integral.

$$\int\limits_{\mathrm{Pic(F)}} e^{sh^0(D)+(1-s)h^1(D)} d[D]$$

In particular, Riemann zeta function is related to the family of ghost-spaces $\mathbb{Z}_u$, where $u(x) = e^{-\pi\alpha x^2}$ for positive α. This extra structure of the ghost-space could be of some interest, as it relates arithmetic to harmonic analysis, which is coherent with some of the recent approaches to the Riemann Hypothesis. For example, the functions of positive type on G are related to the so-called cyclic representations of G (cf. [1], Theorem 3.20). This link deserves to be explored. We leave it to the RH specialists to figure out if it could be of any use.

4) The abstract theory of ghost-spaces, especially its analytic aspects are yet to be fully developed. First of all, one would like to develop the theory of "mixed ghost-spaces" i.e. groups with the convolution structures like in the Example above. One would also like to have a theory which is more symmetric with respect to duality.

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