Minimal Discrepancies of Toric Singularities

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The main purpose of this paper is to prove that minimal discrepancies of n-dimensional toric singularities can accumulate only from above and only to minimal discrepancies of toric singularities of dimension less than n. I also prove that some lower-dimensional minimal discrepancies do appear as such limit.

1. Introduction

First of all let me recall some basic definitions.

Definition 1.1. *A normal algebraic variety X is called $\mathbb{Q}$-Gorenstein if some multiple of the canonical Weil divisor is a Cartier divisor. All varieties in this paper are $\mathbb{Q}$-Gorenstein unless otherwise specified.*

Definition 1.2. *The minimal discrepancy of a variety X is the minimum of the discrepancies of all exceptional divisors of all resolutions of singularities of X.*

Remark 1.1. *The minimal discrepancy only exists when X has log-canonical singularities (see, e.g.* [5] *for the nice introduction). Whenever it exists it is at least -1. One can also talk about log-discrepancies. They are especially useful when the variety has "boundary" (see* [7]*). When there is no boundary they are just the usual discrepancies plus one. Naturally, the log-discrepancies are non-negative for the log-canonical singularities and the minimal log-discrepancy is non-negative if exists. When it is strictly positive, X is said to have only log-terminal singularities. And when the ordinary minimal discrepancy is positive (non-negative) X is said to have terminal (canonical) singularities.*

Remark 1.2. *Of course, the minimal discrepancy of X is a minimum of minimal discrepancies of any affine covering of X. So it should be considered as an invariant of the worst (in certain sense) singularity of X. It's especially interesting because of its role in several "global" conjectures (see* [1], [2]*).*

The basic "local" conjecture on minimal discrepancies is the following one proposed by V. Shokurov.([10])

Conjecture 1.1. *For every natural n minimal discrepancies of n-dimensional log-terminal singularities can accumulate only from above.*

This conjecture in dimension 2 follows rather easily from the classification of 2-dimensional log-terminal singularities. In dimension 3 there is the result of Kawamata ([6]) that the minimal discrepancies of any $3-$dimensional terminal singularity is $\frac{1}{i}$, where i is a positive natural number (in fact, the index of the singularity).

In particular, Shokurov's conjecture implies that for every n there exists a positive constant $\varepsilon(n)$, such that if all discrepancies of n-dimensional variety X are greater than $-\varepsilon(n)$ then they are in fact nonnegative, that is X has at most canonical singularities.

In this paper we will prove the Shokurov's conjecture for a particular case of toric singularities. In fact, our results are much more precise. They are in some sense best possible for nonterminal toric singularities and also very informative for the terminal case. What to do in case of more general singularities is discussed in section 3. The main result of the paper is the following. (Corollary 2.1)

Main Result. For every natural n minimal discrepancies of n-dimensional toric singularities can accumulate only from above and only to minimal discrepancies of toric singularities of dimension less than n.

Remark 1.3. *It can happen that infinitely many different toric singularities have the same minimal discrepancy. I do not consider this as an accumulation of minimal discrepancies. In this case the minimal discrepancy in question does not necessarily come from lower dimension. It may be of the form 1+(minimal discrepancy of lower-dimensional toric singularity). However the only example of that kind I know is the rather trivial case of minimal discrepancy $n-1$ for $2n-$dimensional singularities.*

There is also the result in the opposite direction (see Theorem 2.2) which implies in particular that every minimal discrepancy of a toric singularity of dimension k is a limit of minimal discrepancies of $n-$dimensional toric singularities for n big enough. And it also implies that if this discrepancy is non-positive then the only restriction on n is that $n > k$.

I am glad to thank here V. Shokurov for his constant interest in this study and helpful remarks about it.

2. Proofs

First of all, let me recall some basic facts about toric varieties. (See, for example, [3], [4], [9].) Every n-dimensional affine toric variety X is just a $Spec(R)$, where R is a ring generated by monomials $x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$, where $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is an integral point of some finitely generated convex cone of full dimension $C(X)$ in R^n. For several reasons it is more useful to consider the dual cone $C^*(X)$ in the dual space $V = R^n$. It does not necessarily have dimension n, but we will always assume that it will. Otherwise X would be isomorphic (not canonically) to a product of another toric variety and an algebraic torus of positive dimension.

Now various conditions on singularities of X have simple combinatorial formulation in terms of $C^*(X)$. Namely, let us consider one-dimensional extremal rays $l_1, l_2, ..., l_k$ that generate this cone. They are rational therefore we can pick on every l_i an integral point P_i which is the closest one to zero. Then all P_i lie in one hyperplane if and only if X is $\mathbb{Q}$-Gorensteinian. X is $\mathbb{Q}$-factorial if and only if $k = n$, that is C^* (or C) is simplicial. X is regular if and only if $C^*(X)$ is regular which means that it is simplicial and P_i form a basis for the lattice. Moreover, the Gorenstein index and minimal discrepancy of X also have simple description. Namely, let us consider the linear function F on V, such that $F(P_i) = 1$. (This is possible exactly when X is $\mathbb{Q}$-Gorensteinian.) Then the least common denominator of values of F on non-zero points of C^* is the index. The minimal log-discrepancy which is by definition (1+(minimal discrepancy)) is the minimum of the above values among points in the interior of non-regular sub-cones of $C^*(X)$. (If all sub-cones including $C^*(X)$ itself are regular then X is regular and its minimal discrepancy is undefined.) We will pass freely from discrepancies to log-discrepancies mostly using the latter in proofs and the former in statements.

There is one type of toric singularities which is particularly interesting for our purposes. Namely, quotients of an affine plane A^n by cyclic groups. This corresponds to the case when C^* is simplicial and the lattice N of all integral points in V is generated by P_i and only one extra element x. We can always assume that x lies in the interior of C^* otherwise the singularity splits into the lower-dimensional singularity and torus. I want to mention that the cyclic group in question is just a factor $(N/ < P_i >)$ and its action can be reconstructed from the coordinates of x in the basis $\{P_i\}$. The following lemma reduces everything to this special case.

Lemma 2.1. *The set of minimal discrepancies of toric singularities of dimension n coincides with that of cyclic quotients of dimension no greater than n.*

Proof. Let X, $C^* \subset V$, P_i, F be as above. Let ε be the minimal log-discrepancy of X. By the above combinatorial description there exists an integral point $x \in C^*$, such that $F(x) = \varepsilon$. Consider the ray generated by x. It intersects the polygon $P_1P_2...P_k$ in some point P. Standard combinatorial arguments show that there exists a simplex $P_{i_1}P_{i_2}...P_{i_r} \subset P_1P_2...P_k$, such that P lies in its interior. Of course, this simplex has dimension no greater than $n-1$ and its interior means interior with respect to its own geometry.

Now let us stick to the subspace W of V generated by P_{i_j}. Evidently, $C^* \bigcap W$ will be a convex cone corresponding to some new toric variety X' with the same minimal log-discrepancy ε. This X' is already $\mathbb{Q}$-factorial but it is not a cyclic quotient yet. To produce out of it a cyclic quotient let me do the following. Consider lattice $N' \subset N$ generated by x, change coordinates in such a way that N' become a lattice of integral points and forget about N. What we have now is a cyclic quotient X'', which again has the same minimal log-discrepancy ε, so the lemma is proven.

Remark. As one can easily see from the proof of the above lemma we can assume that factor-group $(N/ < P_i >)$ is not only cyclic but also generated by the element x, which has "minimal log-discrepancy" (that is $F(x) = \varepsilon$.) In the rest of the paper this element will be often called generating element. The fact that it is not uniquely determined for a given singularity does not cause any difficulties.

From now on we stick to this particular case of quotient singularities and whenever we have a toric variety it is a cyclic quotient singularity. The above lemma allows us to do it. Now let me notice that the results we are going to prove are of two types. Most of them are negative in a sense that they restrict where and how minimal discrepancies can accumulate. And there are some positive results based on procedures that allow us to construct cyclic quotients with prescribed minimal discrepancies starting with the given one. We begin with negative results which all deal with the following situation.

Suppose we have a sequence of cyclic quotient singularities $\{X^\nu\}, \nu = 1, 2, ...$, such that their log-discrepancies ε^ν are getting closer and closer to some real number ε. Consider the standard simplex Δ in R^n defined by the inequalities $\alpha_i \geq 0$, $\sum \alpha_i \leq 1$ inside a standard hypercube H, defined by the inequalities $0 \leq \alpha_i \leq 1$. By identifying simplexes $P_1^\nu P_2^\nu ... P_k^\nu$ with this standard one we obtain a sequence of points $\alpha^\nu \in H$ that correspond to x^ν. By the compactness of H there exists a subsequence with a limit point α. We replace our sequence by this subsequence. By the above remark $\varepsilon^\nu = \sum \alpha_i^\nu$. Therefore $\varepsilon = \sum \alpha_i$. Now all our negative results can be formulated as the sequence of statements which will be proven in a row. Let me state this as a theorem.

Theorem 2.1. *In the above notations the following is true.*

1) *If ε^ν are not the same for big ν then α is on the boundary of H.*

2) *If ε^ν are the same for big ν then one can choose α on the boundary of H which has the same ε and is also a limit of some sequence of the same type.*

3) *ε is rational.*

4) *If ε^ν are not the same for big ν they accumulate to ε only from above.*

5) *We can choose α as in 2) on some face of H to be a generating point of a cyclic quotient singularity if considered on this face. As a corollary, ε is a minimal log-discrepancy for some lower-dimensional toric singularity plus some nonnegative integer.*

6) *Every face of H is characterized by restricting some coordinates to be 0 and some coordinates to be 1. Under this remark statement 5 can be strengthen by the restriction that for the face of α the number of $1-s$ is not greater than the number of $0-s$. Moreover if ε^ν are not the same for big ν then the number of $1-s$ is strictly less than the number of $0-s$*

In order to prove this theorem let me introduce the notion of multiple of the point in H. It will be used a lot in the rest of the paper so it deserves to be stated formally.

Definition 2.1. *For every point $\alpha = (\alpha_i) \in H$ and integer m let $m-th$ multiple of α be the point $\alpha^{(m)}$ whose $i-th$ coordinate is 1 if $\alpha_i = 1$ and $\{m\alpha_i\}$ otherwise. Note that for positive m this construction is continuous at the neighborhood of the boundary of H.*

Now we begin the proof.

Statement 1). Suppose α is in the interior of H. Consider $\alpha^{(m)}$ for all integer m. Then the compactness of H tells us that there are two numbers $m_1 < m_2$, such that $\alpha^{(m_i)}$ are very close, for example closer than $\frac{1}{100}\times$ (distance from α to the boundary of H). They may also coincide, we don't care. Then $\alpha^{(m_1-m_2+1)}$

and $\alpha^{(m_2-m_1+1)}$ are evidently very close to $\alpha^{(1)} = \alpha$. We have several cases.

First of all suppose that sum of coordinates of one of the above two points is less than ε (that means that sums of coordinates of $\alpha^{(m_1-m_2+1)}$ and $\alpha^{(m_2-m_1+1)}$ are different.) Let it be $\alpha^{(m_1-m_2+1)}$. Then for ν big enough α^ν is close enough to α and $\alpha^{\nu,(m_1-m_2+1)}$ is close enough to $\alpha^{(m_1-m_2+1)}$ and therefore the sum of coordinates of $\alpha^{\nu,(m_1-m_2+1)}$ is less than ε^ν, which is impossible.

Now suppose that sums of coordinates are the same. Then if there is a subsequence of α^ν for which ε^ν accumulate to ε from above consider $(m_1 - m_2 + 1)-$th multiples. Then for ν big enough from this subsequence sum of the coordinates of $\alpha^{\nu,(m_1-m_2+1)}$ is less then sum of the coordinates of $\alpha^{(m_1-m_2+1)}$, because $(m_1-m_2+1) < 0$. Therefore it is less then ε^ν, which is impossible. Similar arguments work for the case when ε^ν accumulate to ε from below. We should just consider $(m_2-m_1+1)-$th multiples instead of $(m_1 - m_2 + 1)-$th ones and notice that $(m_2 - m_1 + 1) \geq 2$.

Remark 2.1. *We did not prove that point inside H cannot be a limit of generating points of cyclic quotients with the* **same** *discrepancy. And this indeed can happen. The easiest example is given by two-dimensional canonical toric singularities.*

Statement 2). Suppose we have a sequence of points α^ν with the same sum of coordinates ε. Consider those multiples of all these points that have the same sum of coordinates ε. We will see very soon that there are plenty of them. We have two cases.

First of all, suppose α has finite order in H that is $\alpha^{(k)} = 0$ for some k. Then for α^ν that are close enough to α $(mk + 1)-$multiples have sum of the coordinates ε. Moreover when we make m run from zero to some number depending on ν they run following some straight line with small intervals until they hit the boundary of H. The length of these intervals goes to zero when α^ν go to α. Therefore we have infinitely many points in every neighborhood of the boundary of H, intersected with a hyperplane $\sum x_i = \varepsilon$. Therefore there exists a point on this boundary which is a limit of some sequence of these points. To complete the argument it is enough to mention that each one of these points is also a generating point for some quotient singularity with the same discrepancy ε.

Now suppose α has infinite order in H. Nevertheless ε is rational because $\varepsilon = \varepsilon^\nu$. So we have infinitely many multiples of α with the same sum of coordinates ε. Then arguments similar to that of the above case show that whenever two of such multiples are close to each other there is some other multiple that is close to the boundary of H. Again as before, there is a point on the boundary which is a limit of a sequence of these multiples. Now we can just notice that every multiple of α is a limit of multiples of α^ν and the rest is the same as above.

Statement 3). Suppose ε is irrational. By previous statements we can assume that α is on some face of H. Then all its multiples are by the definition on the same face. Now we want to prove that for some $m > 0$ $\alpha^{(m)}$ is close enough to α and sum of the coordinates of $\alpha^{(m)}$ is less than ε. This is not completely trivial, because we require m to be positive. Here is the proof. First of all, we can stick to the face of H α belongs to. Then we notice that all sums of coordinates of $\alpha^{(m)}$ are different because ε is irrational. By the compactness argument there exist some positive integers $m_1 < m_2$ such that $\alpha^{(m_1)}$ and $\alpha^{(m_2)}$ are close enough. If the sum of coordinates of $\alpha^{(m_1)}$ is greater than the sum of coordinates of $\alpha^{(m_2)}$ it is enough to choose m to be equal to $1+m_2-m_1$. Otherwise we need one more step. Denote $m_3 = 1+m_1-m_2$. Then everything would have been OK, but m_3 is not positive. But we can find $m_4 < m_5$ of form $l(m_3+1)$, such that $\alpha^{(m_4)}$ and $\alpha^{(m_5)}$ are so close that $\alpha^{(m_3+m_5-m_4)}$ is still close enough to α and the sum of its coordinates is still less than the sum of coordinates of α.

Now for m as above and ν big enough α^ν is close enough to α therefore $\alpha^{\nu,(m)}$ is close enough to $\alpha^{(m)}$. (Here we really need that m is positive because α lies on the boundary of H.) But this means that for ν big enough sum of the coordinates of $\alpha^{\nu,(m)}$ is less than sum of the coordinates of α^ν, which is impossible.

Statement 4). Now ε is rational. The arguments similar to the above allow us to find an integer $m > 1$ such that $\alpha^{(m)}$ is close to α and has the same sum of coordinates. Namely, the compactness argument tells that there are m_1, m_2 such that $\alpha^{\nu,(m_1)}$ and $\alpha^{\nu,(m_2)}$ are arbitrary close. (They may even coincide, we don't care.) Then $m = 1+m_2-m_1$ will satisfy all requirements.

Now if discrepancies ε^ν accumulate to ε from below then for sufficiently large ν the sum of coordinates of $\alpha^{\nu,(m)}$ is less than the sum of coordinates of α^ν. (By definition m is greater than 1 and $(\alpha^{\nu,(m)} - \alpha^{(m)}) = m(\alpha^\nu - \alpha)$.) This completes the proof of the statement.

Statement 5). We have α on some face of H. Let us consider this face and multiples of α on it. If there are infinitely many of them that are in fact different then there are infinitely many of them with the same sum of the coordinates ε, because ε is rational. Then arguments of the proof of statement 2 allow us to replace α so that it lies on a face of lower dimension. We can do this until we come to α that has finite order in the appropriate face. Now on this face α is a generating point for a quotient singularity, because if some multiple of it has smaller sum of coordinates in the face it has smaller sum of all coordinates and usual arguments show that it is impossible.

Statement 6). Suppose α has order N in its face. Suppose this face is determined by k equalities of type $x_i = 0$ and l equalities of type $x_i = 1$ Consider $(1-N)-$th multiples of α^ν. Then the corresponding $\varepsilon-$s go to $\varepsilon + k - l$ when α^ν go to α. Therefore $k \geq l$. Moreover, if ε^ν accumulate to ε from above then $\varepsilon-$s for $(1-N)-th$ multiples accumulate from below. So case $k = l$ is also impossible.

This completes the proof of the theorem. The following corollary is formally also restrictive but in fact as you can see from its proof it is a positive result (or, more precisely, simple observation.)

Corollary 2.1. *Under the notations of the above theorem if ε^ν are not the same for big ν then ε is not just sum of lower-dimensional log-discrepancy and integer but is a lower-dimensional log-discrepancy itself. If ε^ν are the same for big ν then ε is either a lower-dimensional log-discrepancy or (1+(minimal log-discrepancy of dimension $\leq (n-2)$)).*

Proof. This is a straightforward consequence of statement 6 and the following fact.

Fact. For arbitrary $m-$dimensional cyclic quotient one can construct $(m+2)-$dimensional cyclic quotient whose minimal log-discrepancy is greater than given exactly by 1.

Construction that proves the above fact is as follows. Suppose the generating point α has order N. Then we construct new $(m+2)-$dimensional singularity defined by the generating point $(\alpha, \frac{1}{N}, 1-\frac{1}{N})$ which means that first m coordinates remain the same and last two are as specified.

Remark 2.2. *The main idea of the proof of the above theorem (namely use of multiples, compactness and some sort of continuity) is very similar to that of the boundedness theorem for toric Fano varieties with bounded discrepancies. ([3]) However there everything is written using less geometrical language that maybe hide this idea in formulas.*

Now let me state the most general positive result I know about what lower-dimensional discrepancies can indeed appear as a limit. But before doing this I would like to notice that by the evident reason of symmetry every minimal log-discrepancy of cyclic quotient of dimension n is not greater than $\frac{n}{2}$.

Theorem 2.2. *Suppose we have an $m-$dimensional cyclic quotient generated by α with minimal log-discrepancy ε. Denote $r = -[-\varepsilon]$ so that r is the smallest integer which is greater or equal than ε. Then for all nonnegative integers l, $(\varepsilon + l)$ is a limit of $n-$dimensional log-discrepancies for all $n \geq m + r + 2l$.*

Proof. There are in fact several ways of doing this for nonzero l. The freedom we have is basically due to the Fact in the proof of the above corollary. I will show you just one way.

First of all let me consider the standard $n-$dimensional hypercube H and divide the set of coordinates $\{x_1, x_2 ... x_n\}$ into three parts as follows. First m of them will correspond to the coordinates of our given $m-$dimensional singularity and will be still called x_i. Those with indexes from $m+1$ to $m+l$ will be called $y_i, i = 1, ..., l$. And those with indexes from $m+1+l$ to n will be called $z_i, i = 1, ..., n-m-l$. Now we place our $m-$dimensional singularity on the face of H defined by equalities $y_i = 1, z_i = 0$. Let us denote by T the point $(\alpha; 1, ..., 1; 0, ..., 0)$ that corresponds to the generating point α. Suppose its order is q that is its $q-$th multiple is a vertex of H. Consider vertex $P = (0, ..., 0; 0, ..., 0; 1, .., 1)$. (Here ";" divides x_i, y_i and z_i.) Now the generating points of $n-$dimensional singularities we are looking for lie on the segment PT close to T. More precisely, they are given by formula $(\frac{1}{N}P + (1-\frac{1}{N})T)$ where $q|(N-1)$. In order to complete the proof it is enough to show that for all such points every multiple is either trivial or has sum of coordinates greater or equal than that of the point itself.

So, consider the point $A = A_N$ defined as above. It is a straightforward observation that it has order N. Now arguments similar to those from the proof of the statement 2 of Theorem 2.1 show that for every positive integer $k < N$ k-th multiple of A lies in the set S_k defined by the following procedure.

Procedure. Suppose T_k is a $k-th$ multiple of T. Draw a ray starting from T_k and parallel to the ray $[TP)$. When it hits the boundary of H change the corresponding $1-$s to $0-$s and $0-$s to $1-$s. Do it until the sum of lengths of all segments drawn equals the length of PT.

So it is enough to prove that no point from this set can have sum of coordinates greater than that of A. In order to do it let me make several simple observations. First of all let me notice that in fact when we draw our segments we **never** change 1 to 0, we only change $0-$s to $1-$s. The reason is that in all x_i we draw in the negative direction and we cannot hit the boundary on y or z for $k < N$. Another observation is that "locally", that is when we don't hit boundary, the sum of coordinates does not decrease because sum of coordinates of P is greater or equal than that of T by the condition $n \geq m + r + 2l$. Combined together these two observations evidently take care of k which are not divisible by q. For k divisible by q we only need to notice that we begin our procedure from the point $(0, ..., 0; 0, ..., 0; 0, ..., 0)$ but the first nontrivial segment starts from the point $(1, ..., 1; 0, ..., 0; 0, ..., 0)$ whose sum of coordinates is greater than that of A.

3. Some open questions

There are several natural questions concerning the obtained results.

Question 1. Is it true that EVERY minimal log-discrepancy of $n-$dimensional cyclic quotient is a limit of minimal log-discrepancies of $(n+1)-$dimensional cyclic quotients? The theorem above together with the classification of $3-$dimensional terminal toric singularities implies that this is true for $n \leq 3$. It is natural to try, maybe with computer, the case $n = 4$. As far as I know the classification of $4-$dimensional toric terminal singularities is not yet completed but there are some conjectures and a lot of work is already done. See [8] for details.

Other questions naturally arose when I tried to extend these results to more general singularities.

Question 2. Is it true that the set of minimal discrepancies of quotient singularities with respect to arbitrary groups coincides with that of cyclic quotients of the same dimension?

Question 3. Is there an example of log-terminal singularity whose minimal discrepancy is not a minimal discrepancy for any cyclic quotient of the same dimension?

Question 4. Is it true for arbitrary log-terminal singularities which are not terminal that every (or at least one) divisorial valuation that corresponds to the minimal discrepancy is given by a divisor on the $\mathbb{Q}$-factorial terminal modification in sense of Miles Reid? (Of course it is not true for all valuations with negative discrepancy, but the question is about minimal discrepancy.) It is true for toric singularities and in dimension 2 and I have no counterexamples in the general case.

While stating these questions it would be unfair not to express my opinion about them. I suspect that the answer to the Question 1 is "Yes" for many singularities but not for all of them. The answer to the Question 2 is most probably "Yes". Example to the Question 3 probably also exists, maybe even $3-$dimensional. And I have been unable so far to find any serious evidence pro or against for the Question 4.

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