STATEMENT ON RESEARCH

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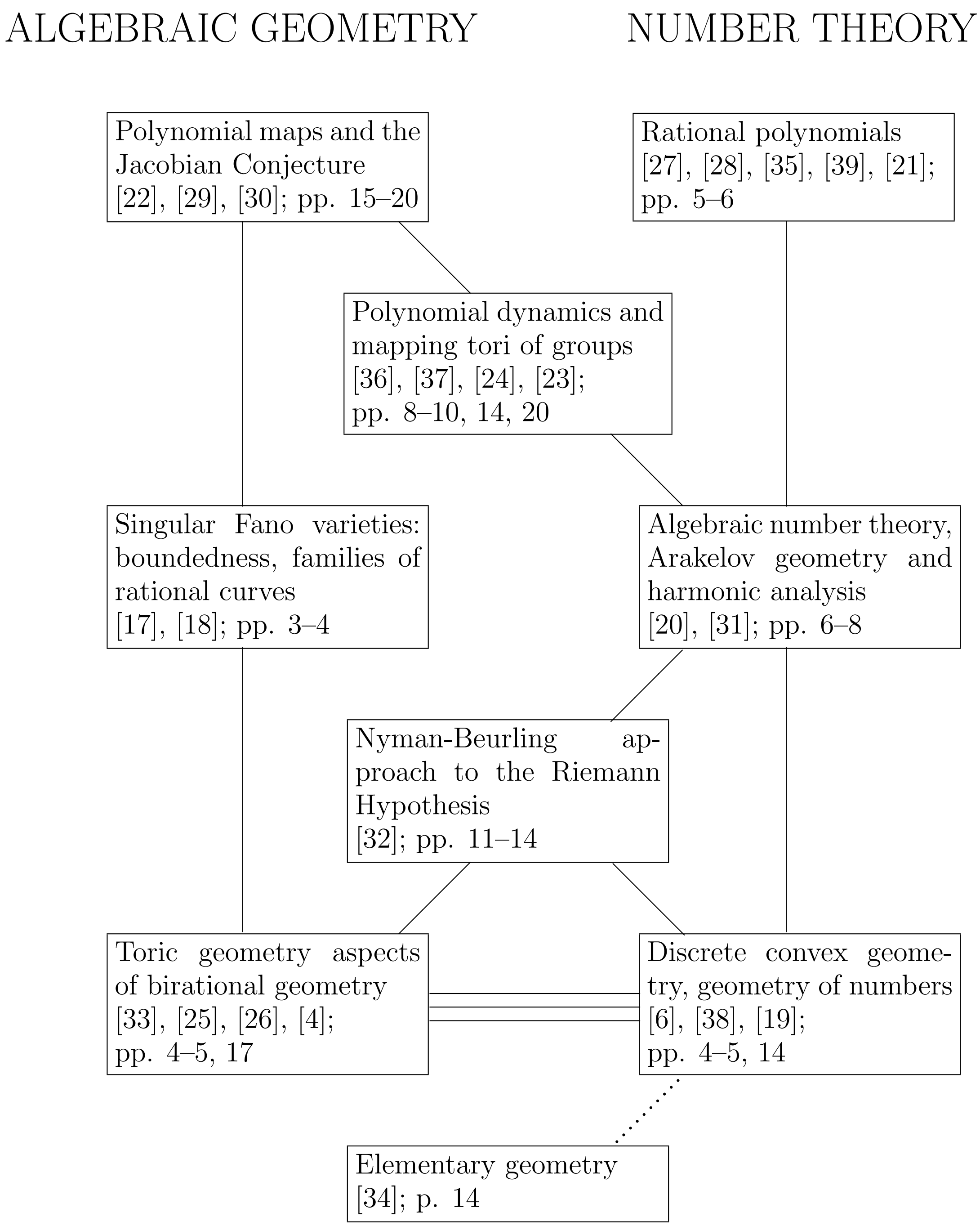
1. Introduction

When it comes to mathematics, I consider myself a geometer, in a broad sense of the word. Essentially, this means that I prefer problems that can be in some sense visualized.

I consider myself a generalist, and strive to discover and explore connections between different branches of mathematics. It is no secret that the majority of serious current research in pure mathematics is highly specialized. As a result, many meaningful and beautiful connections between different areas of mathematics go undiscovered. Whenever one can bridge two different mathematical theories in a meaningful way, exciting things happen. Methods from one area lead to strong and surprising results in the other one, and both areas gain in significance. Sometimes, a totally new research area is created, at the crossroads of two or more formerly independent subjects. I have been very successful in discovering and developing these connections. I am also proud of my more specialized research, especially in birational algebraic geometry.

In this research statement I will first describe my older research, and then the research I have done in the last decade or so, while at the University of Pittsburgh and at Binghamton University. The chart on the next page is intended to show the different areas of mathematics I have worked on, indicating some of the connections. The numbers in each box refer to the relevant papers and to the pages of this research statement, where they are discussed. Generally, the algebraic geometry research is on the left, and the number theory research is on the right. The triple connection between toric geometry aspects of birational geometry and discrete convex geometry indicates that these are, essentially, the same subject. The dotted line connecting the elementary geometry to the convex discrete geometry indicates a very weak connection.


ALGEBRAIC GEOMETRY
NUMBER THEORY
Polynomial maps and the Jacobian Conjecture [22], [29], [30]; pp. 15–20
Rational polynomials [27], [28], [35], [39], [21]; pp. 5–6
Polynomial dynamics and mapping tori of groups [36], [37], [24], [23]; pp. 8–10, 14, 20
Singular Fano varieties: boundedness, families of rational curves [17], [18]; pp. 3–4
Algebraic number theory, Arakelov geometry and harmonic analysis [20], [31]; pp. 6–8
Nyman-Beurling approach to the Riemann Hypothesis [32]; pp. 11–14
Toric geometry aspects of birational geometry [33], [25], [26], [4]; pp. 4–5, 17
Discrete convex geometry, geometry of numbers [6], [38], [19]; pp. 4–5, 14
Elementary geometry [34]; p. 14

2. Older Research

I started my research career as a birational algebraic geometer. My birational geometry research started with my thesis, which was concerned with Fano varieties with log-terminal singularities.

Definition. A Fano variety is a projective variety with ample anti-canonical class. That is, some tensor power of the top exterior power of the tangent bundle has enough sections to embed X into a projective space. Note that for singular varieties the anti-canonical class is only defined as a Weil divisor. Being ample implies that some multiple of it is a Cartier divisor. The smallest natural m, such that mK_X is Cartier, is called the index of X.

Definition. A variety X has log-terminal singularities if for any resolution of singularities $\pi : Y \to X$ all coefficients a_i in the adjunction formula $K_Y = \pi^* K_X + \sum a_i E_i$ are greater than -1. Here E_i are the exceptional divisors of π; the numbers a_i+1 are called log-discrepancies.

In my thesis ([17]) I proved boundedness of Fano varieties of dimension three of given index, subject to two technical simplifying assumptions: they are $\mathbb{Q}$-factorial, with Picard number one (in other words, unipolar). I was later able to remove these technical restrictions ([18]). This means that these Fano varieties belong to a finite number of families. This result is the three-dimensional case of a conjecture of Batyrev, which in turn is a small part of the following very strong boundedness conjecture for Fano varieties.

Borisov-Alexeev-Borisov Conjecture. For any fixed n and positive ε, Fano varieties with log-terminal singularities with log-discrepancy greater than (or equal to) ε belong to finitely many families.

This conjecture was proposed independently by me and by Valery Alexeev, and received considerable attention. (The second Borisov in its name is my brother Lev; our joint paper [33] settled this conjecture in the toric case). It has several implications in higher-dimensional algebraic geometry. Many renowned algebraic geometers worked on it, including Alexeev, Batyrev, Clemens, Kawamata, Kollár, M^{c}Kernan, Miyaoka, Mori, Nikulin, Takagi and Ran (cf. [3], [51], [57], [79]). Still, my 2001 result stayed as one of the strongest for a long time. It was only relatively recently surpassed by Hacon, M^{c}Kernan and Xu who, in a very important paper, proved, among other results, the Batyrev Conjecture in any dimension (cf. [46])). Before that, my result was an ingredient in the proof by Kollár, Miyaoka, Mori and Takagi of boundedness of three-dimensional Fano varieties with canonical singularities ([57]).

In a surprising recent development, the full Borisov-Alexeev-Borisov conjecture has been proved by Caucher Birkar ([14]). This summer, he was awarded the Fields medal for this and related work.

The general idea of my approach was to use the rational curve techniques of Kollár-Miyaoka-Mori ([55], [56]) to obtain a curve with small intersection with the anticanonical class, and then use the general boundedness result of Kollár. The work was done primarily on the terminal modification of the Fano variety, which is a partial resolution of singularities, with relatively numerically effective canonical class and terminal singularities. Serious difficulties occur when one tries to glue the free family of curves with the curves on some exceptional divisor on the terminal modification, in the case when these curves on the divisor pass through the terminal singularities of the ambient variety. In this case one needs to resolve these singularities, while still keeping under control the degrees of the curves involved. This was done by an application of a variation of the Minimal Model Program due to Alexeev ([2]). The proof is rather delicate, and the main ideas are quite technical.

The one class of varieties for which BAB Conjecture was originally proven is toric Fano varieties. Toric Fano varieties correspond to convex lattice polytopes P that contain the origin. The log-discrepancy is greater than ε if and only if $(\varepsilon \cdot P) \cap \mathbb{Z}^n = \{0\}$. Together with my brother Lev we proved that in any fixed dimension there are only finitely many such polytopes, up to lattice isomorphisms ([33]). This was our first paper, we did this research while still being undergraduates at Moscow State University. Unfortunately, the main part of our proof was not as new as we thought at the time. As I learned later from Jeff Lagarias, this discrete geometry statement was first proven by Hensley ([47]), and improved upon by Lagarias and Ziegler ([59]). However, this research led me to a long-term interest in discrete convex geometry, specifically the problem of classification of convex lattice polytopes and cones with few or no lattice points inside. My strongest result in this direction is that in any fixed dimension the lattice simplices with no lattice points inside form finitely many "families"[1] ([26]). This means that at least theoretically all toric terminal singularities of fixed dimension can be classified, which is a qualitative generalization to arbitrary dimension of the Terminal Lemma of Morrison-Stevens ([69], also cf. White, [86]) and the conjectural classification of four-dimensional prime quotients of Mori-Morrison-Morrison ([67]). Together with the result of Sankaran

[1]These are not the families in the algebraic geometry sense, as toric singularities are rigid. One example of such "family" is the DuVal singularities A_n.

this implies that the classification of cyclic quotient singularities prime index of [67] is complete, up to possibly a finite number of exceptions. Unfortunately, the theorem is not effective, so it is not possible at this time to claim that the classification of [67] is actually complete.

In fact, the ultimate result of [26] is even stronger, with the terminality condition being replaced by log-discrepancy being greater than any fixed positive ε. I also proved the toric case of a conjecture of Shokurov on accumulation of minimal log-discrepancies ([25]). This discrete convex geometry research of mine led to a joint paper with Shokurov (cf. [38]).

Together with Lev we also obtained, back in 1989-90, an explicit classification of three-dimensional convex lattice polytopes containing the origin, with no other lattice points in the interior or on the faces and edges. We also found all lattice simplices containing the origin and having no other lattice points in the interior (but possibly points on the boundary). These unpublished results later became the starting point for Al Kasprzyk's thesis work ([52], [53]). In 1999, I put together a little survey of the results in the area (cf. [19]). Despite all of the successes, I was not particularly fond of this part of my research, considering it relatively easy and inconsequential, until I discovered that it is deeply connected to the Nyman-Beurling-Báez-Duarte reformulation of the Riemann Hypothesis, to be described in the next section.

My number theory research started with two papers on irreducibility of polynomials. The original motivation came from trying to mimic the Oesterlé's proof of the ABC theorem of Stothers in the case of integers, using the so-called quantum deformation of integers ($[n]_x = x^{n-1} + \ldots + x + 1$). This naturally lead to the following integer polynomials, for any coprime triple of natural numbers (a, b, c) with $a = b + c$:

$$f_{abc}(x) = \frac{bx^a - ax^b + c}{(x-1)^2}.$$

The corresponding splitting field is unramified over $\mathbb{Q}$ outside of the support of abc. While there is no reason to expect that any of these polynomials are reducible, their irreducibility is very hard to prove. I had to combine information about the distribution of their roots in complex numbers and all $p-$adic complex numbers for $p|abc$ to show that most of them, in the density sense, are irreducible ([28]). This research led me to Michael Filaseta, probably the world's best specialist on irreducibility of rational polynomials. It turned out that he has considered before, with T.-Y. Lam, some related polynomials, the higher order derivatives of $x^{n-1} + \ldots + x + 1$. We ended up proving some strong irreducibility results for them. The method was similar to that of [28], but the final proof was very subtle and complicated ([35]). I still don't know much about my polynomials f_{abc} and whether or not they are indeed related to the *abc* conjecture of Masser-Oesterlé. They did reappear several years later, related to some harder than intended problem posed by Joe Harris in his Harvard algebraic geometry course. I don't think this led to any publications, but several people worked on that problem for a couple of months, including Izzet Coskun and Jason Starr, with the most beautiful final proof obtained by Noam Elkies. A version of this problem was included in the 2014 Putnam Examination as the problem A5. It is fair to say that the Putnam contestants confirmed the difficulty of this problem, see the statistics here: http://kskedlaya.org/putnam-archive/putnam2014stats.html.

Later, I published another paper on quantum integers, joint with Nathanson and Wang ([39]). Unfortunately, that paper contained a mistake, corrected in [21]. In yet another paper on rational polynomials, somewhat related in spirit to the others, I proved that a polynomial with rational coefficients divides the derivative of a polynomial with only rational roots if and only if all of its irrational roots are real and simple ([27]). The argument is a modification of a celebrated theorem of Belyi, the main idea is the similarity between the formula for the derivative of $(x-a_1)^{m_1} \cdot \ldots \cdot (x-a_k)^{m_k}$ and the Lagrange interpolation formula.

My first "interdisciplinary" research was my paper on arithmetic cohomology ([20]). This is my contribution to the fundamental problem of understanding the similarity between the rings of polynomials and integers, and thus between objects of algebraic geometry and number theory.

One of the most famous theories in arithmetic geometry is Arakelov geometry, which provides concrete meaning to the statement "Number Theory at infinity is Analysis." Many of the constructions of classical algebraic geometry can be extended to the Arakelov geometry setting. In particular, an analog L of a divisor on a complete curve is a fractional ideal in some number field together with a choice of real constants for all Archimedean valuations that correspond to the orders of zeroes/poles "at infinity". Naively, the space $H^0(L)$ of global sections of a line bundle is the (finite) set of elements in the corresponding fractional ideal with Archimedean valuations bounded depending on the given corresponding constants. One then defines the "dimension" of this space by taking logarithm of the number of elements.

With this naive definition, one gets classical Riemann-Roch formula, $h^0(L) - h^0(K - L) = \deg L - \frac{1}{2} \deg K$, but only approximately (Here K is the relative canonical class, with 0 constants at infinity). A much better approach, stemming from Tate's thesis and finalized by van der Geer and Schoof ([44]) is to count all elements in the corresponding fractional ideal, but with the contribution decaying as a normal distribution, with parameters depending on the constants at the Archimedean valuations. This way one gets the exact Riemann-Roch formula, using Poisson summation formula.

There was no doubt that van der Geer and Schoof got the dimension of the space of global sections of the Arakelov line bundle right, and it resulted in interesting further development, in particular the paper of Lagarias and Rains ([58]). However, the space of the global sections itself was not really defined. And there was no definition whatsoever of H^1, the first cohomology space. Instead, they just defined $h^1(L)$ as $h^0(K-L)$. In my paper I managed to remedy this and develop a theory totally analogous to the theory of algebraic curves, including defining $H^1(L)$ by a procedure similar to the Čech cohomology. The Tate's Riemann-Roch theorem was separated into the "modern" Riemann-Roch theorem

$$h^0(L) - h^1(L) = \deg L - \frac{1}{2} \deg K$$

and Serre's duality $\widehat{H^1(L)} = H^0(K - L)$. (Thus $h^1(L) = h^0(K - L)$).

Of course, there was a price to pay for this. In order to develop this kind of theory one has to abandon the category of the abelian groups.

This was done in [20] by considering a new kind of objects that truly belong to harmonic analysis, namely locally compact abelian groups with convolution of measures structures of a particular kind. In [20] these objects were named the ghost-spaces. Formally, a ghost-space is a triple (G, u, μ) where G is a locally compact abelian group, u and μ are a function and a measure on G that satisfy certain properties. Intuitively, to construct the ghost-space G_u^μ one starts with the group G and then makes its elements "partially existent" and/or "imprecisely positioned". The corresponding convolution of measures structure on a group G is such that for all $x, y \in G$

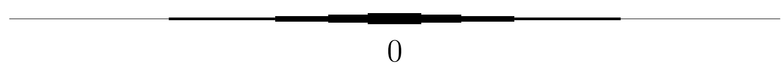
$$\delta_x * \delta_y = \frac{u(x)u(y)}{u(x+y)} T_{x+y}\mu$$

Here u is the "effectivity function" that measures the extent to which the elements exist, and μ is a probability measure which is responsible for the ambiguity of the position of the elements. The $T_{x+y}\mu$ is the translation of μ by the element $(x+y)$.

The corresponding ghost-space is denoted by G_u^μ. When $u \equiv 1$ and $\mu = \delta_0$ then the corresponding convolution structure is just the standard convolution structure on G. Therefore it makes sense to omit the point measure δ_0 and/or the identity function 1 from the notation of ghost-spaces when possible. That is, $G_u \equiv G_u^{\delta_0}$, $G^\mu \equiv G_1^\mu$, $G \equiv G_1^{\delta_0}$.

The following picture represents the ghost-space $\mathbb{R}_{e^{-\pi x^2}}$. One should think of it as being embedded into the usual real line:

Ghost-space $\mathbb{R}_{e^{-\pi x^2}}$



The quotient $\mathbb{R}/\mathbb{R}_{e^{-\pi x^2}}$ is the space $\mathbb{R}^\mu$, where μ is the probability measure $e^{-\pi x^2}dx$. In fact, the short exact sequence of ghost-spaces

$$0 \to \mathbb{R}_{e^{-\pi x^2}} \to \mathbb{R} \to \mathbb{R}^{e^{-\pi x^2}dx} \to 0$$

is the $\mathbb{R}$-analog of the short exact sequence of locally compact groups

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

where $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are the groups of the p-adic integers and the p-adic numbers respectively.

Perhaps the most remarkable aspect of this theory is that the arithmetic analog of Serre's duality turned out to be the Pontryagin duality of the convolution of measures structures. It is closely related to the Pontryagin duality theory of M. Rösler (cf. [80], [81]). One basically gets the perfect theory in this one-dimensional case. There is also an adelic version of this theory, due to Ichiro Miyada (unpublished). I tried to develop the theory further, and was able to settle some harmonic analysis issues, that naturally appeared (cf. [31]). However, my attempts to extend the theory to arithmetic surfaces have been unsuccessful. For a long time I was convinced that the correct framework for this generalization would be some similar kind of spaces with the underlining space being the Parshin-Beilinson's higher-dimensional adeles (cf. [8], [9], [73], [74]), but there are some obstacles that at this time I cannot resolve. Most recently, I came back to this topic and have been pushing it further, with my graduate students, Patrick Milano (graduated this May, [64]) and Changwei Zhou (expected to graduate in 2019). I should also mention the recent paper of Thomas McMurray Price ([63]).

Another instance of this "interdisciplinary" work is my collaboration with Mark Sapir, in which we developed an unexpected, yet natural, connection between algebraic geometry and group theory ([36], [37]). To describe this research, let me first recall some definitions and fix the notation.

Let G be a group given by generators $x_1, ..., x_k$ and a set of defining relations R, and let $\phi\colon x_i \mapsto w_i$, $1 \leq i \leq k$, extend to an injective endomorphism of G. Then the group

$$HNN_\phi(G) = < x_1, ..., x_k, t \mid R, txt^{-1} = w_i, i = 1, ..., k >$$

is called the *mapping torus* of ϕ (or *ascending HNN extension of G corresponding to* ϕ). This group has an easy geometric interpretation as the fundamental group of the mapping torus of the standard 2-complex of G with bounding maps the identity and ϕ. The simplest and one of the most important cases is when G is the free group F_k of rank k, i.e. when R is empty.

A group is called residually finite if the intersection of its normal subgroups of finite index is trivial. It is a natural question to ask whether all mapping tori of free group endomorphisms are residually finite (see, e.g. [66]). In particular, many of the groups of the form $\mathrm{HNN}_\phi(F_k)$ are hyperbolic (see [11] and [50]). One of the outstanding open problems about the hyperbolic groups is to determine whether or not they are all residually finite (see, e.g. [88]).

Residual finiteness of the mapping tori of free group endomorphisms is exactly what we proved in [36]. The first idea is to associate to ϕ a family of self-maps of some affine algebraic varieties over finite fields. Specifically, let H be any group or a group scheme. Then one can define a map $\phi_H\colon H^k \to H^k$ that takes every k-tuple $(h_1, ..., h_k)$ to the k-tuple

$$(w_1(h_1, ..., h_k), w_2(h_1, ..., h_k), \dots, w_k(h_1, ..., h_k)).$$

Notice that this map is not a homomorphism. Nevertheless, it defines a dynamical system on H^k. A particularly interesting case of this general construction is when $H = \mathrm{SL}_r(F_q)$ for some finite field F_q. Using adjunction instead of inversion, this map can be extended to a self-map Φ of $(M_r)^k$, where M_r is the variety of all $r \times r$ matrices. It turned out that periodic orbits of this map Φ for different F_q are directly related to the residual finiteness of $T = HNN_\phi(G)$.

In general, periodic orbits of algebraic maps are very hard to study. In characteristic 0 there are usually very few of them (cf. [42]). However, over the finite fields one can expect to have many periodic orbits. It is still very hard to deal with them in general, but there are some orbits that are easier to study. These are the orbits which consist of points conjugate over the base field. In the language of schemes these orbits correspond to the closed scheme points that are fixed by Φ.

Following [36], one can make the following definition.

Definition. Suppose $\Phi\colon X \to X$ is a self-map of a variety over a finite field $\mathbb{F}_q$. A geometric point x of X over some finite extension of $\mathbb{F}_q$ is called quasi-fixed with respect to Φ iff $\Phi(x) = \mathrm{Fr}^m(x)$. Here Fr^m is the m-th composition power of the geometric Frobenius morphism.

Quasi-fixed points appeared in the Deligne Conjecture on Lefschetz Trace Formula and its generalizations, that wer studied before by Fujiwara and Pink and others (see, e.g., [43], [78]). However these investigations were mostly limited to the quasi-finite maps and most of the maps Φ are not quasi-finite.

If X is the affine space, the above definition becomes the following. Let $\Phi\colon A^n(\mathbb{F}_q) \to A^n(\mathbb{F}_q)$ be a polynomial map, defined over the finite field $\mathbb{F}_q$. It is given in coordinates by the polynomials $\phi_1, ..., \phi_n$ from $\mathbb{F}_q[x_1, ..., x_n]$. Suppose a point $a = (a_1, a_2, ..., a_n) \in A^n$ is defined over the algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. It is a quasi-fixed point of Φ if and only if for some $Q = q^m$ for all i

$$\phi_i(a_1, a_2, ..., a_n) = a_i^Q.$$

Here is the main theorem regarding such maps (see [36]).

Theorem. Let $\Phi^n\colon A^n(\mathbb{F}_q) \to A^n(\mathbb{F}_q)$ be the n-th iteration of Φ. Let V be the Zariski closure of $\Phi^n(A^n)$. Then the following holds.

(1) All quasi-fixed points of Φ belong to V.
(2) Quasi-fixed points of Φ are Zariski dense in V. In other words, suppose $W \subset V$ is a proper Zariski closed subvariety of V. Then for some $Q = q^m$ there is a point $(a_1, ..., a_n) \in U \setminus W$ such that for all i $f_i(a_1, ...a_n) = a_i^Q$.

The first statement is rather easy. The proof of the second statement in [36] is much trickier, but it is essentially elementary.

A stronger result was independently and virtually simultaneously obtained by E. Hrushovski .

Theorem. (Hrushovski, [48]) Let $\Phi\colon X \to X$ be a dominant self-map of an absolutely irreducible variety over a finite field. Then the set of the quasi-fixed points of Φ is Zariski dense in X.

The second part of our theorem follows from Hrushovski's theorem applied to $V = \Phi^n(A^n)$. Hrushovski's proof is very complicated. Besides the standard algebraic geometry techniques like étale cohomology and intersection theory, he also uses some mathematical logic, in particular his theory of difference schemes. As Hrushovski notes, it would be nice to prove his results entirely by the algebraic geometry methods. There are many indications that this is possible. In particular, much of the proof of our theorem works for arbitrary varieties. Also, more recent results on the Deligne Conjecture may be helpful (cf. e.g. [85])

Hrushovski's theorem allowed us to generalize our result as follows (cf. [36]).

Theorem. The mapping torus of any injective endomorphism of a finitely generated linear group is residually finite.

Later, Mark Sapir and I returned to this research and strengthened our result. This will be discussed in the next section.

3. More recent Work and Future Plans

Right before I came to the University of Pittsburgh, I discovered an amazing connection between toric singularities and the Nyman-Beurling approach to the celebrated Riemann Hypothesis (cf. [32]). I stumbled upon it somewhat by accident, by noticing that some results of Vasyunin on the Báez-Duarte's version of the Nyman-Beurling criterion bear striking similarity to the results of Mori-Morrison-Morrison and Sankaran on four-dimensional cyclic quotient singularities. This similarity runs very deep, at the heart of both investigations lie the same inequalities involving the integer part function. The same inequalities also appear in the discrete geometry of the lattice-free simplices (this was already well known to birational geometers) and integer ratios of factorials, a classical area, with intermittent research going back more than a hundred years ([60], [75], [76], [77]).

First, let me describe the Nyman-Beurling -Báez-Duarte's reformulation of the Riemann Hypothesis.

Theorem (Báez-Duarte, [5])

Consider the space V of sequences $\{f : \mathbb{N} \to \mathbb{R}\}$ with the norm defined by the formula

$$||f||^2 = \sum_{n=1}^{\infty} \frac{|f(n)|^2}{n(n+1)}$$

For each integer $k \geq 2$ consider the sequence $f_k(n) = k\{\frac{n}{k}\}$, the remainder of n, when divided by k. Denote by W the closure of the span of $f_k, k = 2, 3, ...$ Then the RH is equivalent to $W = V$, which is in turn equivalent to W containing the constant sequence 1.

Note that traditionally instead of sequences one considers step-functions from $[1, \infty)$ to $\mathbb{R}$, that is functions which are constant on all intervals $[n, n+1)$. When one tries to improve on a known approximation of 1, one is interested in finite linear combinations of f_k that have bounded values everywhere, so that correcting at one place does not cause too much trouble elsewhere. This led Vasyunin to look for linear combinations of f_k that only take values 0 and 1.

The main object of the Mori-Morrison-Morrison's paper [67] is the 4-dimensional cyclic quotient singularities of prime order. Specifically, if p is a prime number, and a_1, a_2, a_3, a_4 are integers, then the cyclic quotient singularity of type $\frac{1}{p}(a_1, a_2, a_3, a_4)$ is the quotient of the affine space A^4 by the action γ of the group μ_p of $p-$th roots of unity, defined as follows: for every $z \in \mu_p$ and a point $x = (x_1, x_2, x_3, x_4) \in A^4$, $\gamma(z)(x) = (z^{a_1}x_1, z^{a_2}x_2, z^{a_3}x_3, z^{a_4}x_4)$. Among these singularities Mori, Morrison and Morrison were looking for so-called terminal singularities.

Definition. An algebraic variety X has terminal singularities if and only if for one (or any) resolution of its singularities $\pi : Y \to X$ all coefficients a_i in the adjunction formula $K_Y = \pi^*(K_X) + \sum a_i E_i$ are positive.

In the case of cyclic quotient singularities $\frac{1}{p}(a_1, a_2, a_3, a_4)$ this condition can be described as follows. Consider the simplex $\Delta \subset \mathbb{R}^4$, defined by inequalities $x_i \geq 0, i = 1, 2, 3, 4; x_1 + x_2 + x_3 + x_4 \leq 1$. Enlarge the standard lattice $\mathbb{Z}^4$ by adjoining the point $\frac{1}{p}(a_1, a_2, a_3, a_4)$. The singularity is terminal if and only if the only points in this new lattice in Δ are still the vertices of Δ.

The following example illustrates the connection that I discovered.

Example. For every n and k,

$$\frac{(2n)!(2k)!}{n!k!(n+k)!} \in \mathbb{Z}$$

These numbers go back to Catalan, but their combinatorial interpretation is still unclear, except for small k (cf. [45]). One way to prove that they are integers is to calculate the power of an arbitrary prime p in them. By a well known formula for the powers of primes in factorials, this amounts to proving that

$$\sum_{i=1}^{\infty}([\frac{2n}{p^i}] + [\frac{2k}{p^i}] - [\frac{n}{p^i}] - [\frac{k}{p^i}] - [\frac{n+k}{p^i}]) \geq 0.$$

This follows from the following inequality for the integer part (floor) function:

$$[2\alpha] + [2\beta] - [\alpha] - [\beta] - [\alpha + \beta] \geq 0$$

In fact, the above number is an integer, and is either 0 or 1. The same inequality allowed Mori, Morrison and Morrison to prove, in particular, that the cyclic quotient singularities $\frac{1}{p}(2a, 2b, -a, -b)$ are terminal. On the Nyman-Beurling side, this corresponds to some five-term linear combinations of step-functions that only take values 0 and 1, discovered by Vasyunin.

When my paper [32], revealing the connection between these formerly unrelated subjects, first appeared as a preprint, it led to a flurry of activity. First, Jonathan Bober proved a conjectural classification of Vasyunin, which generalized a conjecture of Mori-Morrison-Morrison, proven by Sankaran (cf. [15]). This was achieved by relating the corresponding integer ratios of factorials to algebraic hypergeometric functions, following the idea of Fernando Rodriguez Villegas. For instance, one of the 29 "stable quintuples" of Mori-Morrison-Morrison (one of the 29 five-term step functions of Vasyunin) corresponds to the integer ratios of factorials $\frac{(9n)!n!}{(5n)!(3n)!(2n)!}$. The generating function $\sum_{n=0}^{\infty} \frac{(9n)!n!}{(5n)!(3n)!(2n)!}x^n$ is an algebraic hypergeometric function. These functions were previously classified by Beukers and Heckmann ([12]).

Then independently and virtually simultaneously Bell and Bober, and Bombieri and Bourgain (two Fields medalists!), proved a vast qualitative generalization of the Vasyunin's conjecture, proposed by me (cf. [10], [16]). As a result, a new area of research has been created, related to algebraic and convex geometry, analysis and analytic number theory.

Many open questions and challenges remain, of which possibly the most exciting is to understand the algebraic structure behind the Nyman-Beurling-Báez-Duarte reformulation of the Riemann Hypothesis. Some glimpses of this structure include the dilation operators and a convolution structure on the space of the step-functions (closely related to the Dirichlet convolution, and ultimately to the multiplication of Dirichlet series). Because the Nyman-Beurlling approach has been pursued almost exclusively by analysts and analytic number theorists, I feel that there is a lot more algebra behind it than what has been discovered so far. Of course, there is no doubt that if the Riemann Hypothesis is ever proved or disproved using this approach, the analysis will play the pivotal role. I have been also playing with the idea that the integer part function should be replaced by a theta function, in the same way as the naive count of global sections of an Arakelov line bundle is just a crude approximation to the correct count that uses the normal distribution to count all elements of the fractional ideal with different weights. If successful, this may lead to a criterion for the Riemann Hypothesis in terms of linear combinations of modular functions (of different level). While these are much more complicated objects than sequences, they have been extensively studied, and this approach would have an advantage of openly addressing the zeta-function's symmetry about the critical line, an important piece of the puzzle that gets totally hidden in the Nyman-Beurling approach. Also, the results of [10] and [16] imply a higher-dimensional Terminal Lemma for 1-parameter families of cyclic quotients, but they do not apply to the "exceptional" ones. A stronger conjecture of mine, which is a proper higher-dimensional generalization of the Terminal Lemma of [69], is still open but seems to be within reach.

Most recently, together with Barile, Bernardi, and Kantor, we proved that all four-dimensional toric singularities are cyclic quotients ([6]). This brings further significance to the classification of [67]. Together with main results of [26] and [15] this almost proves the completeness of the classification of four-dimensional lattice-free simplices. To complete the proof of the classification, more work needs to be done. Ideally, one should make effective the argument of Jim Lawrence ([61]).

Following up on our Inventiones paper, Mark Sapir and I proved the following theorem.

Theorem.([37]) All mapping tori of free group endomorphisms are virtually residually (finite p-groups), for all large enough p. (This means that for every mapping torus for every large enough prime p there is a subgroup of finite index such that the intersection of all its subgroups $p-$power index is trivial).

This is a much stronger result than the residual finiteness theorem of [36]. Basically, we had to construct more finite quotients with a much better control of their structure. The idea of the proof is to lift the quasi-fixed points over the finite fields to the recurrent points over local fields (in the $p-$adic topology). The proof is quite technical, and took long time to carefully write up, because some results related to the algebraic maps over local fields have not been written up, to our knowledge, in the generality that we needed. With more technical work, the proof can most probably be extended to arbitrary finitely generated linear groups using the theorem of Hrushovski.

Perhaps my most elementary paper up to date is the joint work with Stuart Hastings and Mark Dickinson that was published in Monthly ([34]). This paper was selected by MAA to receive the Ford Award for mathematical exposition. The highlight of this research for me was the discovery that one can define a square by just four distance and angle measurements, instead of five that are needed for a generic quadrilateral.[2] This sort of questions is directly related to the second order stability of tensegrity networks. Even though it can hardly be classified as serious mathematics, it appeals to the geometer in me, and I take it as evidence that many beautiful mathematical truths are hiding in plain view.

My latest big research project is a geometric approach to the famous Jacobian Conjecture of Keller, in dimension two ([22]). This 80-year old conjecture states that any polynomial map of (two-dimensional) complex affine space which is locally 1-to-1 is globally invertible. This conjecture has got a well-deserved reputation of being much trickier than one might expect. Indeed, many "proofs" of it were claimed and then rescinded, by respectable mathematicians. This situation is in sharp contrast with the recent advances in birational geometry. During the past thirty years our understanding of the structure of the higher-dimensional algebraic varieties has grown dramatically, and many seemingly harder questions have been answered. Moreover, much research has been done on rationally connected varieties and rational curves on them, including the work of Keel and M^{c}Kernan on rational curves on quasi-projective surfaces ([54]). Many deep results related to the Jacobian Conjecture have been obtained by the specialists in polynomial automorphisms, including the celebrated Abhyankar-Moh theorem ([1]) and the theorem of Shestakov and Umirbaev that the Nagata automorphism is not tame ([83]). While many researchers tried to approach this conjecture geometrically, including, in particular, Miyanishi (cf., e.g. [49]), Domrina and Orevkov ([41], [40]), Le Dung Tráng ([62]), most approaches have been algebraic in nature.

My approach can be described as a birational geometer's approach. Suppose a counterexample to the Jacobian Conjecture exists. It gives a rational map from P^2 to P^2. After a sequence of blowups of points, we can get a surface X with two maps: $\pi : X \to P^2$ (projection onto the origin P^2) and $\phi : X \to P^2$ (the lift of the original rational map).

Note that X contains a Zariski open subset isomorphic to A^2 and its complement, $\pi^*((\infty))$, is a tree of smooth rational curves. We will call these curves exceptional, or curves at infinity. The common way to deal with them is by means of a (dual) graph Γ. Its set of vertices is the set of the exceptional curves, with two vertices connected by an edge whenever the curves intersect. The vertices are usually labeled by the self-intersections of the corresponding curves. The structure of

[2]The square ABCD of side 1 is the only quadrilateral with $|AB| = |AD| = 1$, $|AC| = \sqrt{2}$ and $\angle BCD = \frac{\pi}{2}$

this graph is easy to understand inductively, as it is built from a single curve (∞) on P^2 by a sequence of two operations: blowing up a point on one of the curves or blowing up a point of intersection of two curves. However, a non-inductive description is probably impossible, which is the first difficulty in this approach. Another difficulty comes from the fact that the exceptional curves on X may behave very differently with respect to the map ϕ. More precisely, there are four types of curves E.

type 1) $\phi(E) = (\infty)$

type 2) $\phi(E)$ is a point on (∞)

type 3) $\phi(E)$ is a curve, different from (∞)

type 4) $\phi(E)$ is a point not on (∞)

From a first glance, the situation appears almost hopelessly complicated. However I managed to show ([22]) that it is a lot more orderly than one may expect. In particular, for a given graph of curves, one can essentially always tell which curves are of which type, and there is a fairly restrictive family of graphs that can potentially appear in a counterexample to the JC.

The first idea is to introduce a new labeling on the graph Γ of exceptional curves as follows. By induction, the classes of the exceptional curves E_i form an integer basis in the Picard group of X. Define the augmented canonical class of X, $\bar{K_X} = K_X + \sum E_i$, a very natural object from the viewpoint of the Minimal Model Program. It is equal to $\sum a_i E_i$, where a_i are integers. We label E_i by these coefficients a_i. The main advantage of this labeling is that, unlike the traditional self-intersection labels, these labels do not change under additional blow-ups. The label only depends on the divisorial valuation that the exceptional curve defines.

One can easily see that when a point of intersection of two curves is blown up, the new curve is labeled by the sum of the labels of its two "parent" curves. When a new point is blown up on a curve, the label of the exceptional curve is one plus the label of its "parent". At the beginning of the blowup process we have a single curve with label (-2). By induction, one can easily see the following.

1) The labels of any two adjacent vertices in Γ (i.e. intersecting exceptional curves on X) are coprime.

2) The set of vertices with negative labels form a connected subgraph of Γ.

3) This subgraph is separated from the "positive" part of Γ by vertices with label 0. Each if these 0-vertices can only have (-1)-vertices or 1-vertices as neighbors.

The defining property of any counterexample to the Jacobian Conjecture is that all ramification of the map $\phi : X \to P^2$ is in the exceptional curves. Because of this, any curve of type 1 must have a negative even label, while every curve of type 3 must have a positive label. The subgraph of all curves of type 1 and 2 is connected, and every curve of type 3 is adjacent to it, while any curve of type 4 is not. We can assume that the last curves blown up, when creating X from P^2, are of type 1 or 3, and this allows us to determine for a given graph the curves of type 3.

One can see easily that some curves of type 3 must be present in any counterexample to JC. One can prove that $\pi_*^{-1}(\infty)$ is curve of type 2. A more in-depth investigation, using some inequalities reminiscent of those that I used in [17], allows to show that the subgraph of the curves of type 2 must be connected. This implies in particular that they are all mapped to the same point by ϕ. As a result, the graph Γ has the following structure. It has a connected subgraph of type 2 curves, which includes $\pi_*^{-1}(\infty)$. Adjacent to it are curves of type 1 and type 3, and "behind" them are curves of types 2 and 4 respectively. One can go even further and prove that there are no curves of type 2 "behind" the curves of type 1. The main ingredient in the proof is an observation that the "di-critical log ramification divisor" on the Stein factorization of ϕ, which is the sum of curves of type 3 with ramification indices, must be ample ([29]).

One more interesting labeling can be defined on Γ. For a curve E_i, consider the determinant of the graph matrix (= minus-intersection matrix) on all other curves. Like the canonical labeling, this determinant label only depends on E_i, and not on any additional blowups. It is positive if and only if all the other curves are simultaneously contractible in the analytic category. One can prove that any curve of type 1 must have a negative determinant label, which provides further restrictions on the graph Γ.

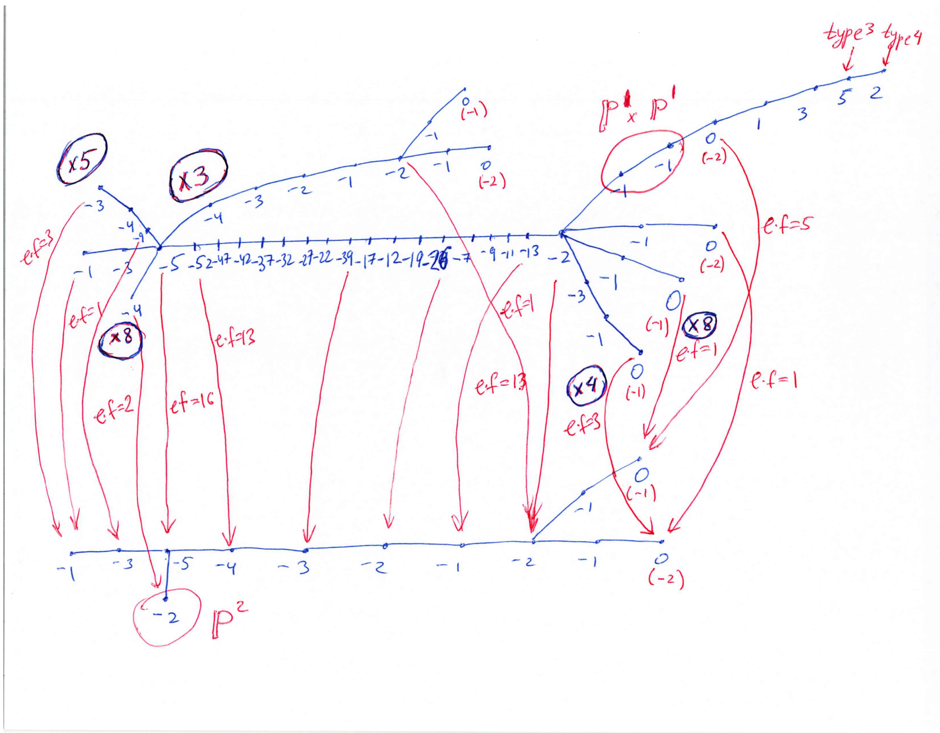
These two labelings of curves, being invariant under subsequent blowups, are invariants of the corresponding divisorial valuations. They are also invariant under the polynomial automorphisms of the plane. I was recently able to show that whenever these two invariants are fixed, the corresponding divisorial valuations form a finite number of families up to the polynomial automorphisms. The main idea is that the determinant labels can be tracked during the blowups if one also keeps track of the "determinant labels of edges", i.e. the determinants of the graph obtained by removing the corresponding edge. The proof of the main theorem is purely combinatorial and rather complicated. Perhaps a more conceptual proof exists. I was also able to show using these invariants that all curves of type 1 lie in the same connected component of the graph obtained from the full graph of exceptional curves by removing the strict pullback of the line at infinity (cf. [30]).

Absence of ramification in the plane also gives very strong restrictions on the singularities of the middle surface in the Stein factorization of ϕ, especially after a suitable sequence of blowups at the target results in the images curves of type 3 intersecting the curves at infinity transversally. In particular, all these singularities are cyclic quotients. This suggests that one may want to look for the counterexamples to the Jacobian Conjecture by going through singular surfaces obtained by a sequence of weighted blowups. The advantage of this approach is that one can potentially look much further, and the only maps one needs to consider are finite, as opposed to generically finite maps.

Last January, I managed to construct two families of compactifications of the affine plane that seem to support a counterexample to the Jacobian conjecture: they satisfy all restrictions coming from the Picard group. If true, the counterexample would have topological degree 16 and would be given by polynomials of degree 99 and 66. This pair of degrees was actually the last tricky case considered and discounted by Moh ([65]). However, Moh's proof in this case is sketchy. In a still unpublished 2016 preprint Moh's former student Yansong Xu alleged a gap in Moh's work, and a fix for it ([87]). Both arguments are too complicated for me to verify. My own extensive computer calculations (using Maple, involving several hundreds of variables) also seem to imply that no such Keller map can exist. But at this time I do not have a **simple** reason for that. To give you an idea of the complexity of this, here is the schematic picture of the associated graphs, with some explanations.

The figure below shows the graphs of the curves at infinity on X and Y. The vertices are labeled by their $\bar{K}$ labels, and whenever the $\bar{K}$ label is 0, the number in parentheses below it is the self-intersection. (Note that for all other curves the self-intersection can be recovered from the $\bar{K}$ labels by the adjunction formula). To keep from overcrowding, is is not indicated where some of the curves go, but this can be mostly guessed from the picture. For instance, the curves with $\bar{K}$ labels $-47, -42, -37, -32, -27, -22$ all go to the intersection of the curves with $\bar{K}$ labels -4 and -3. The curves with the $\bar{K}$ labels 1 and 3 go to a point on the curve with $\bar{K}$ label 0 and self-intersection (-1). This is the point of intersection of this curve on Y with the image of the curve with $\bar{K}$ label 5 (it is a curve of type 3). The curve with $\bar{K}$ label 2 goes to some point in $\mathbb{A}^2$.

Framework Example



The notation $e\cdot f = 13$ indicates that the product of the ramification index e and the degree of the restriction to the curve f equals 13; note that this local degree is constant along the branches. The notation x5 indicates that there are five branches like that; in the case of x3 the entire branch with the fork is produced three times (and each branch is mapped to the branch with the fork downstairs one-to-one). For the curves of type 1 the $\bar{K}$ label on X equals the product of the $\bar{K}$ label on Y and the ramification index e. The surface Y is constructed from $\mathbb{P}^2$ and the surface X is constructed from $\mathbb{P}^1 \times \mathbb{P}^1$, as indicated on the picture. Importantly, I calculated exactly the push-forwards and pull-backs of all these curves, and these operations satisfy all required properties, including the projection formula.

One can recognize in the above figure two rational Belyi maps: above the curve with $\bar{K}$ label -5 and the curve with the $\bar{K}$ label -2. These maps can be explicitly found using computer. For instance, the first one can be given, up to a change of variables, by the rational map $w \mapsto \frac{p^2(w)}{w\cdot r^3(w)}$, where the polynomials $p(w)$ and $r(w)$ are the following:

$$p(w) = w^8+(2+8\sqrt{-3})w^7+\frac{-233+50\sqrt{-3}}{3}w^6+\frac{-4600-376\sqrt{-3}}{3}w^5+$$
$$\frac{835-890\sqrt{-3}}{3}w^4+\frac{2420+22\sqrt{-3}}{3}w^3+(\frac{1043}{3}+336\sqrt{-3})w^2+$$
$$(-118+158\sqrt{-3})w+(-28+41\sqrt{-3}),$$

$$r(w) = w^5+\frac{4+16\sqrt{-3}}{3}w^4+\frac{-278+68\sqrt{-3}}{9}w^3+(-\frac{140}{3}-24\sqrt{-3})w^2+$$
$$\frac{35-112\sqrt{-3}}{3}w+\frac{68-20\sqrt{-3}}{3}.$$

Note that $deg(P^2 - w\cdot r^3) = 3$.

A lot more can be done. For each edge of the graph on Y and X one can explicitly write down some rational functions that give a pair of local coordinates on that "cross". Then one can figure out various degrees, in particular the degrees of the two-variable polynomials that would give a Keller map that would match the above framework. This is how we get the degree 99 and 66 (more specifically, 72+27 and 54+18 if we count separately the degrees for the input variables). There are some fairly strong restrictions on what coefficients can appear and also many linear relations among these coefficients. Note that by virtue of these relations we go much further than just the investigation of the Newton Polygons.

Overall, I am very optimistic about my approach. I feel that it is very natural and may eventually help solve the Jacobian conjecture in dimension two. I have been at it for more than a decade, but, while the progress is not fast, I have never had a feeling of hitting a wall. Interestingly, I still do not really "know" whether the conjecture is true or false, but whatever the answer is, I am getting closer and closer to it.

Working on the Jacobian Conjecture also made me come back to my birational geometry roots. In particular, together with Valery Alexeev we proved a toric case of the following conjecture of M^{c}Kernan (which is a part of a more general conjecture of Shokurov): for every Mori contraction $f : X \to Y$ with ε-log terminal X, the resulting Y is δ-log terminal, with δ depending only on ε and the dimensions of X and Y. This conjecture is related to the Borisov-Alexeev-Borisov conjecture, and several interesting new questions naturally arise, both in the toric and in the general case (cf. [4]).

I also recently wrote a couple of papers on the dynamics of polynomial maps over finite fields (cf. [24], [23]). There I build on some ideas that came up in the joint work with Sapir to construct examples of polynomial maps with some unusual properties.

Example ([24]) Define an integer polynomial map $F(x, y)$ by the formula

$$F(x, y) = (x^2y, x^2y + xy^2)$$

Then F and all of its reduction modulo primes p are dominant. However all points over $\mathbb{Z}/(p\mathbb{Z})$ are sent to $(0, 0)$ by some iteration of F_{at}.

Example ([23]) Suppose F is a finite field, and $a \in F$. Define the map T from the $X = A^3(F)$ to itself as follows:

$$T(x, y, z) = \big((x^2 + az^2)(x - y)z^3, ((y^2 + az^2)^2 + az^4)(x - y)z, (x - y)z^5\big)$$

Define $Y \subset X$ by the equation $x^2+az^2 = yz$. Then Y is "geometrically nilpotent, but not nilpotent" with respect to T: every geometric point of Y over any finite field is mapped to the fixed point $(0, 0, 0)$ by some iteration of T, but no single iteration works for all points.

These and similar examples lead to several interesting open questions, some of them possibly very deep.

One advantage of having broad research interests is that I have come across, and developed some intuition about, a substantial number of open questions and projects of different levels of difficulty and technical sophistication. Obviously, I have had other interests besides the topics on which I published, for example the Lehmer Conjecture on the Mahler measure. The breadth of my interests makes it easy to include graduate students in research, and I found this to be beneficial for everybody involved. In particular, I have been directing two of my Ph.D. students, Patrick Milano (graduated this May, [64]) and Changwei Zhou to expand on my work on Arithmetic Cohomology ([20], [31]). I feel that Arakelov Geometry is due for a new conceptual breakthrough, that may lead to exciting developments.

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