Projective Geometry Approach to the Jacobian Conjecture

Alexander Borisov

Binghamton University

October 22, 2020

ZAG Seminar

Jacobian Conjecture. Every locally invertible polynomial self-map $F : \mathbb{C}^n \to \mathbb{C}^n$ is globally invertible.

Some History. The conjecture goes back to a 1939 paper by Ott-Heinrich Keller. The name "Jacobian Conjecture" was coined by in the 1970s (Abhyankar, Moh?). The conjecture attracted a lot of attention, but is still wide open. It is quite infamous for a large number of proposed solutions by respectable mathematicians. There are many partial results.

Elementary Remarks.

1) Locally invertible $\iff$ Jacobian not equal to 0, where the Jacobian for a self-map given by the polynomails $f_1(x_1, ..., x_n), ..., f_n(x_1, ..., x_n)$ is $\det(\frac{\partial f_i}{\partial x_j})$.

2) Because the Jacobian of f is in $\mathbb{C}[x_1, ..., x_n]$ it is only invertible if it is a non-zero constant.
($\emptyset$ in Abhyankar's notation).

3) Dimension 1 case is obvious.

Basic Results

1) (F is injective) $\Longrightarrow$ (F is sujective)
(Ax-Grothendieck Theorem)
2) (F is invertible) $\Longrightarrow$ (F^{-1} is a polynomial map)

Some Further Results

1) Bass, Connell, Wright (1982): JC is true iff it is true for polynomials of degree at most 3.

(Drużkowski, (1983): polynomials of the form
$f_i = x_i + H_i^3$, where H_i are linear.)

2) Wang, 1980: JC is true for polynomials of degree at most 2.

. . .

Literally, there have been hundreds of papers on JC, with various algebraic, geometric, topological approaches. Just listing them would take a whole hour, and a different speaker.

Some Dimension 2 Results

Abhyankar (1970s) JC is true if the field extension is Galois, in particular, degree 2

Orevkov (1996) degree 3

Domrina, Orevkov (1998) degree 4

Żoladek (2008) degree 5; also degree $2p$, prime p

Moh (1983) JC is true for pairs (f_1, f_2) of degrees up to 100*.

Many other restrictions on the possible counter-examples to JC (**Keller maps**) have been proven. Again, I am not the right person to give a survey talk on these results.

*A complicated paper, with some proofs only sketched

Weak Jacobian Conjecture

(Surjectivity Conjecture)

$Jac(F) = \emptyset \implies F$ is surjective

This makes sense even in *char* p. Particular case: conjecture of Vasiu ($\approx$ 2010)

Conjecture.

Suppose $F = (f_i)$, where $f_i = x_i + g_(x_1, ..., x_n)^p$. Then F is surjective.

This conjecture is open even for $n = 2$, over F_p^{alg}.

My personal guess: JC is false in higher dimensions, possibly even in dimension 2. Surjectivity Conjecture may be true in general; likely true in dimension 2.

Rationale:

1) Luroth Problem

Cubic hypersurfaces are quite different from quadratic hypersurfaces

2) Shestakov, Umirbaev (2002): Nagata automorphism is not tame.

Main (Obvious) Difficulty

F is not proper. It is only quasi-finite, not finite. (Otherwise, we would have a covering of C^n, thus a bijection).

Nice Example

$$F(x,y) = (x^2y + x + y, xy)$$

F is quasi-finite of degree 2, ramified at a smooth rational curve of degree 3. $Im(F) = \mathbb{C}^2 \setminus \{(0,-1)\}$.

Natural Approach: compactify, resolve

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\ \mathbb{P}^2 = X \arrow[rr, dashed] & & Y = \mathbb{P}^2 \end{tikzcd}$$

Here π is birational, a sequence of blowups "at infinity", i.e. outside of $\mathbb{A}^2$. And ϕ is generically finite.

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\phi"] \arrow[r, "\tau"] & W \arrow[d, "\rho"] \\ \mathbb{P}^2 = X \arrow[rr, dashed] & & Y = \mathbb{P}^2 \end{tikzcd}$$

Here τ is birational, ρ is finite, W is normal, algebraic (Stein factorization).

$$\begin{array}{ccc} & Z \xrightarrow{\tau} W & \\ \pi \swarrow & \searrow \phi \quad \downarrow \rho & \\ \mathbb{P}^2 = X & - - - - - \rightarrow & Y = \mathbb{P}^2 \end{array}$$

Structure of Z

$Z = \mathbb{A}^2 \sqcup (\cup E_i)$, where E_i are "curves at infinity"

Among E_i we have $\pi_*^{-1}(\infty)$. $Pic(Z)$ is freely generated by the classes of E_i.

With respect to ϕ, there are **4 types of E_i**:

type 1) $\phi(E)$ is a curve, $\phi(E) \cap \mathbb{A}^2 = \emptyset$
(i.e., $\phi(E) = (\infty)$)

type 2) $\phi(E)$ is a point outside of $\mathbb{A}^2$

type 3) $\phi(E)$ is a curve, $\phi(E) \cap \mathbb{A}^2 \neq \emptyset$
(i.e., $\phi(E) \neq (\infty)$)

type 4) $\phi(E)$ is a point in $\mathbb{A}^2$

Following Orevkov, we call curves of type 3 dicritical divisors.

Note: τ contracts all curves of type 2 and 4, so on W we only have curves of type 1 and 3.

$$\begin{array}{ccc} & Z \xrightarrow{\tau} W & \\ {}^{\pi}\swarrow & \searrow^{\phi} \downarrow^{\rho} & \\ \mathbb{P}^2 = X & - - - - - \rightarrow & Y = \mathbb{P}^2 \end{array}$$

Augmented Canonical Class

$\bar{K}_Z = K_Z + \sum E_i$, $\bar{K}_W = K_W + \sum\limits_{type\ 1,3} E_i$

$\bar{K}_X = -2L_X$, $\bar{K}_Y = -2L_Y$

Adjunction Formula

$\bar{K}_W = \rho^* \bar{K}_Y + \bar{R}$, where $\bar{R} = \sum\limits_{type\ 3} r_i R_i$ is the "dicritical log-ramification divisor": $R_i = \tau(E_i)$ and r_i is the ramification index of ρ at R_i.

Note: Adjunction Formula encodes both fundamental properties of the Keller map: $\mathbb{A}^2$ goes to $\mathbb{A}^2$ and all ramification is outside of $\mathbb{A}^2$.

The structure of Z is largely determined by the weighted graph (tree) of the curves at infinity. One problem: self-intersections are not invariants of the divisorial valuations.

Definition. For a curve E_i its $\bar{K}$ label is the coefficient a_i in $\bar{K} = \sum a_i E_i$

$$\begin{tikzcd}
 & Z \arrow[r, "\tau"] \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & W \arrow[d, "\rho"] \\
\mathbb{P}^2 = X \arrow[rr, dashed] & & Y = \mathbb{P}^2
\end{tikzcd}$$

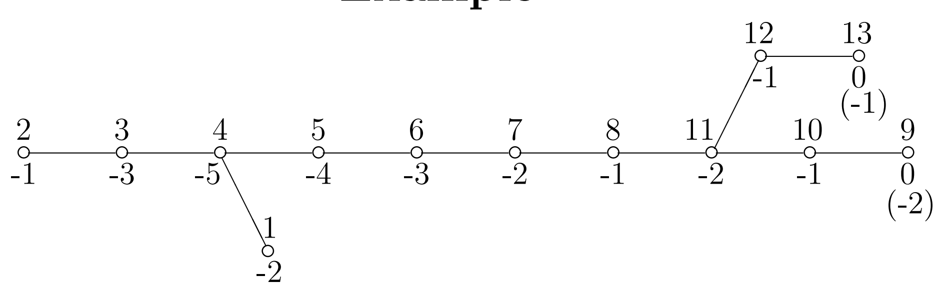
Adjunction Formula implies that all curves E_i of type 1 have negative $\bar{K}$ labels and all curves of type 3 have positive $\bar{K}$ labels: r_i.

Rules: 1) $(\infty) \subset \mathbb{P}^2$ has $\bar{K}$ label -2.

2) When a point on E_i is blown up, the new curve gets $\bar{K}$ label $a_i + 1$.

3) When $E_i \cap E_j$ is blown up, the new curve gets $\bar{K}$ label $a_i + a_j$.

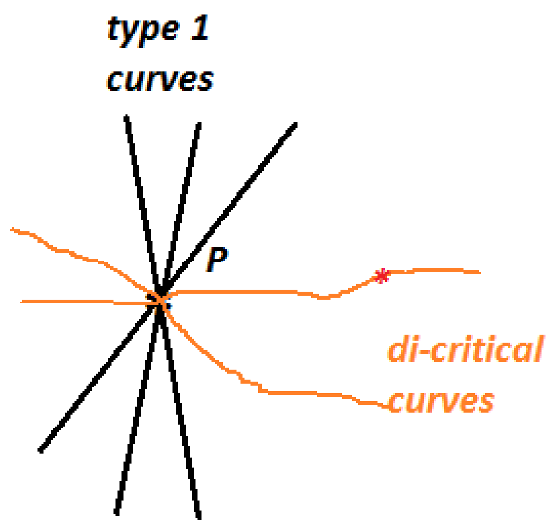
Example



The numbers below are $\bar{K}$ labels; above indicate order of creation. In parentheses: self-intersections.

$$\begin{tikzcd} & Z \arrow[r, "\tau"] \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & W \arrow[d, "\rho"] \\ \mathbb{P}^2 = X \arrow[rr, dashed] & & Y = \mathbb{P}^2 \end{tikzcd}$$

Structure of W (A.B., 2015) $\bar{R}$ is ample, moreover:



$P = \tau(\pi_*^{-1}(\infty))$

Idea of the Proof. $\bar{K}_W = \rho^* \bar{K}_Y + \bar{R}$

$-2 < R_i \cdot \bar{K}_W = \rho_*(R_i) \cdot \bar{K}_Y + R_i \cdot \bar{R} \leq -2 + R_i \cdot \bar{R}$

So $R_i \cdot \bar{R} > 0$. Thus $\bar{R}^2 > 0$. So $(\tau^* \bar{R})^2 > 0$, thus $Supp(\tau^* \bar{R}) \ni \pi_*^{-1}(\infty)$. So $\pi_*^{-1}(\infty)$ is of type 2.

$$\begin{array}{ccccc} & & Z & \xrightarrow{\tau} & W \\ & \swarrow^{\pi} & & \searrow^{\phi} & \downarrow \rho \\ \mathbb{P}^2 = X & - & - & - \to & Y \to \mathbb{P}^2 \end{array}$$

What's next?

Modify Y by blowing up $\phi(\pi_*^{-1}(\infty))$ until it becomes a curve.

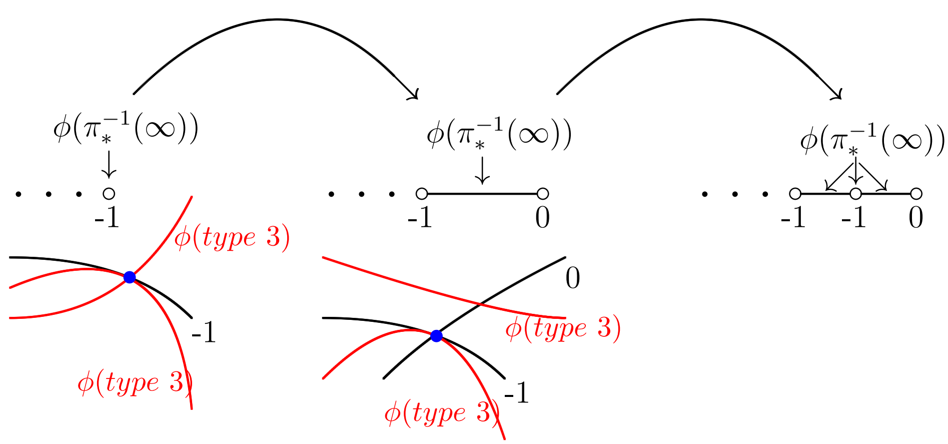
For any Y the 4 types of curves at infinity on Z still make sense and the only type change that may occur is $2 \to 1$.

Properties of the process

1) The tree of curves on Y only gets vertices of valency 1, 2, or 3.

2) All curves on Y have non-positive $\bar{K}$ labels; $\bar{K}-0$ curves have valency 1.

3) While $\phi(\pi_*^{-1}(\infty))$ is a point, all $\phi(R_i)$ must contain it or intersect only a $\bar{K}-0$ curve.



$$\begin{array}{ccccc} & & Z & \xrightarrow{\tau} & W \\ & \overset{\pi}{\swarrow} & & \overset{\phi}{\searrow} & \downarrow \rho \\ \mathbb{P}^2 = X & - & - & - \dashrightarrow & Y \to \mathbb{P}^2 \end{array}$$

4) All curves on Y have non-negative *determinant labels*

Definition. The determinant label of a curve F_i in $Y \setminus \mathbb{A}^2$ is the corresponding principal minor of the Gram matrix of minus-self-intersection form.

Theorem. (A.B., 2014) Divisorial valuations at infinity with given $\bar{K}$ and determinant labels are bounded up to polynomial automorphisms.

5) Mori Cone on Y is spanned by curves at infinity.

6) Above curves with valency 1 the map ϕ is 1-to-1. Above curves with valency 2, it is $x \mapsto x^n$. Above curves with valency 3, we get rational Belyi maps.

Informally, the boundary of a Keller map is a tree of rational Belyi maps.

Question. Can one construct "a map between trees of curves at infinity" of compactifications of $\mathbb{A}^2$ that satisfies all the conditions of a Keller map?

Answer. Yes! (A.B., 2020)

First Framework

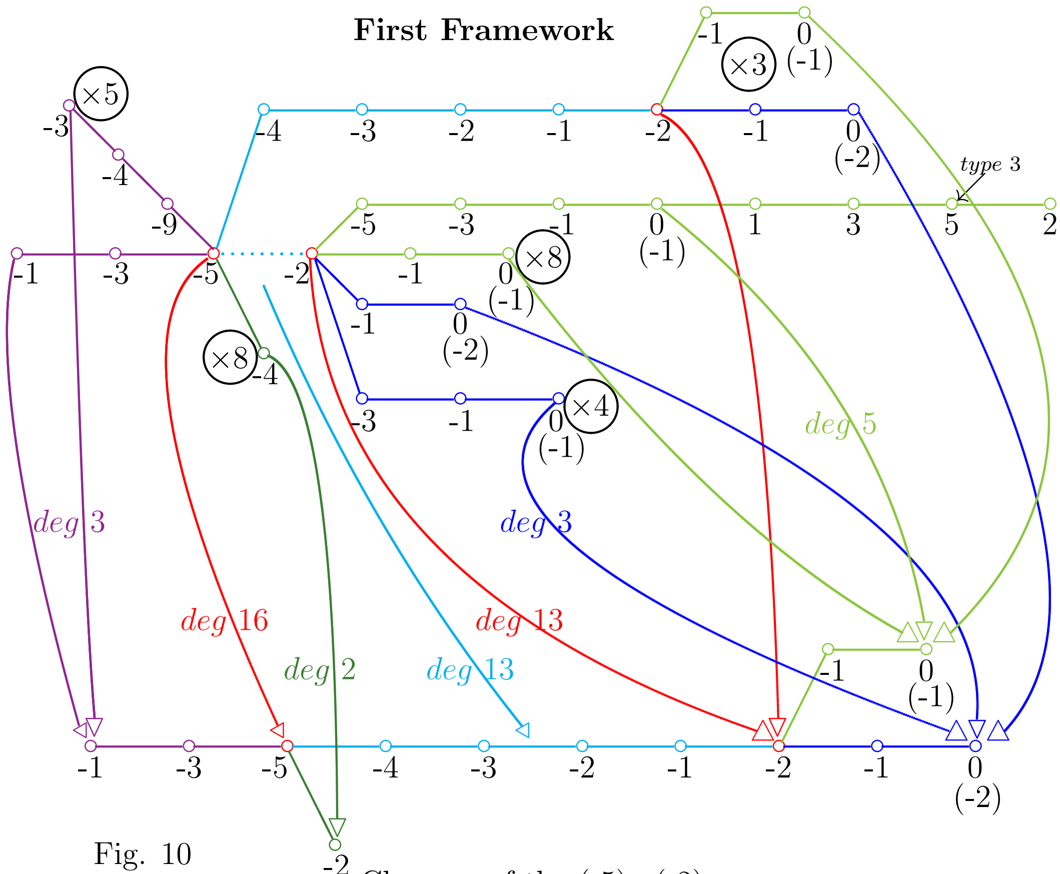
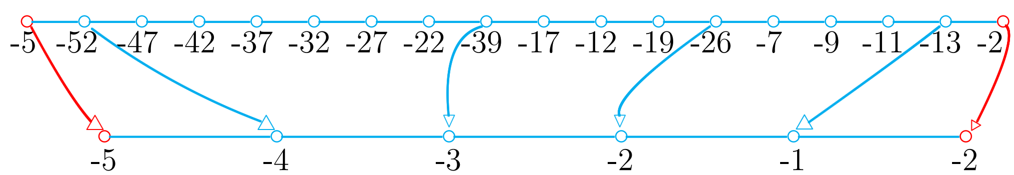


Fig. 10

Close-up of the (-5)...(-2) map



This actually comes with ϕ^* and ϕ_* that satisfy the projection formula. This framework corresponds to a hypothetical Keller map with polynomials of degrees 99 and 66, and degree of field extension 16.

Each framework leads to a system of polynomial equations on the coefficients of the two coordinate functions and the structural parameters of the surfaces Z and Y. My calculations show that in the above framework there is no actual map, but I can not completely trust my bookkeeping.

This (99,66) is the last case in Moh's "<100" paper, but the proof there is only sketched. And there is some controversy about it. It would be nice to have a simple **geometric** reason for why there is no Keller map like this.

What's next?

In my calculations there were "no miracles". So the next step is to learn to exactly calculate the number of variables and the number of "independent" equations for any framework. Then try to construct a framework with more variables than equations. If successful, look for a Keller map. If unsuccessful, try to figure out why. As you see, I still do not know if JC in dimension 2 is true.

Collaborators Wanted!

References

1) A. Borisov. On the Stein factorization of resolutions of two-dimensional Keller maps. *Beitr. Algebra Geom.* **56** (2015), no. 1, 299–312.

2) A. Borisov. On two invariants of divisorial valuations at infinity. *J. Algebraic Combin.* **39** (2014), no. 3, 691–710.

3) A. Borisov. Frameworks for Keller Maps. *Electronic Journal of Combinatorics* **27(3)** (2020), 3.54.

Available at

http://people.math.binghamton.edu/borisov/research.html

Thank You!