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Lattice simplices with few points inside: the birational geometry connection

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Algebraic Geometry's goal: classification of all algebraic varieties…

say, over $\mathbb{C}$, irreducible

Dimension 1: (curves)

Normal (=smooth), Complete (=projective) curves	1-to-1 $\longleftrightarrow$	field extensions of $\mathbb{C}$ of tr.deg 1

Genus *g* = "number of handles"

Canonical class *K* = "divisor of differential form"

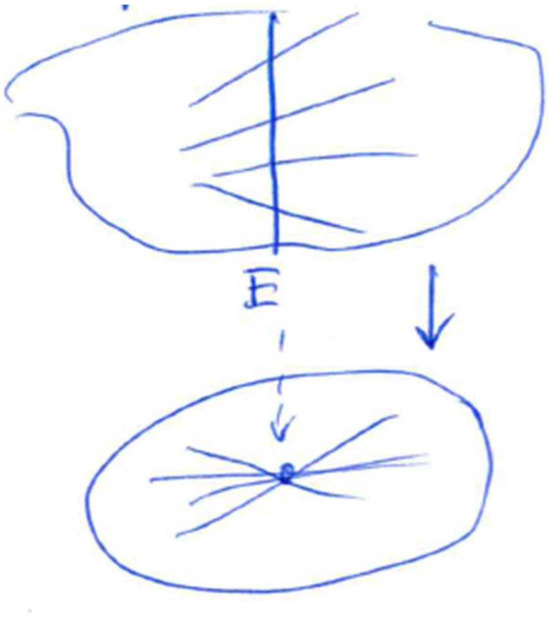
deg K = 2g-2

Depending on genus:

Genus	Name	Canonical class	Universal cover
$g=0$	$\mathbb{P}^1$	-*K* ample	$\mathbb{CP}^1$
$g=1$	elliptic curves	*K=0*	$\mathbb{C}$
$g\geq 2$	general type	*K* ample	$\mathbb{H}$ (upper half plane)

Dimension 2: (surfaces)

New phenomenon: blow-up, blow-down



smooth

smooth

E is (-1)-curve

$$E^2 = -1, \quad K \cdot E = -1, \quad E \cong \mathbb{P}^1$$

$$2\, g_{\tilde{E}} - 2 \leq E^2 + K \cdot E$$

= iff E is smooth

Classical Minimal Model Theory of surfaces

$$\begin{tikzcd} S \arrow[r, "\text{blow-down}"] & S_1 \arrow[r] & S_2 \arrow[r] & \ldots \arrow[r] & S_N \end{tikzcd}$$

smooth
projective

no (-1) curves
(minimal model)

Classification of surfaces: Zoo!

Barth – Hulek – Peters – van de Ven   Compact complex surfaces

Kodaira dimension: rate of growth of $h^0(nk)$

$-\infty$	$\mathbb{P}^2$, F_n  $E^2 = -n, n = 0,2,3,\cdots;$ ruled (Hirzebruch surfaces)
0	Abelian, $K3$, Enriques, bi-elliptic
1	elliptic
2	general type

What about non-smooth surfaces? Normalization has isolated singularities. Then there is "minimal resolution"

<u>Higher dimensions</u>

Hironaka, 1964: Resolution of singularities always exists (char 0):

$$\begin{tikzcd} Y \arrow[d, "\text{generically 1-to-1 (birational)}"] & \text{smooth} \\ X & \text{any variety} \end{tikzcd}$$

Minimal Model Program, $dim \geq 3$

If $\exists\, \underset{\substack{\uparrow \\ \textit{curve}}}{C}$ s.t. $C \cdot K < 0$, ∃ a contraction $\varphi : X \longrightarrow X_1$

Three Types

Fibration

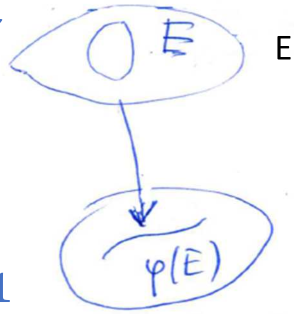
$$\begin{tikzcd} X \arrow[d, "\varphi"] \\ Y \end{tikzcd}$$

dim Y<dim X

STOP!

Divisorial contraction

$$\begin{tikzcd} X \arrow[d] \\ X_1 \end{tikzcd}$$



E is codim 1

dim $\varphi(E)$ < *dim* E

X_1 may have singularities

(fairly mild)

KEEP GOING!

Small contraction

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & \hat{X} \arrow[dl] \\ & X_1 & \end{tikzcd}$$

dim (exceptional set)

is ≤ dim X-2

Make a flip,

KEEP GOING!

Main problems:

1) Existence of flips

2) Termination of flips

Singularities in MMP

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \\ X \end{tikzcd}$$

smooth, exceptional divisors in simple normal crossings

$\mathbb{Q}$-Gorenstein

discrepancies

$$K_Y = \pi^* K_X + \Sigma\, a_i\, E_i$$

(1+ a_i is log-discrepancy)

		dim 2	*dim 3*	*dim 4, higher*
$\forall a_i > 0$	terminal	=smooth	cyclic quotients of smooth or CDV singularities Reid, 83; Mori, 85	$\boxed{?}$ Mori, Morrison Morrison,88
$\forall a_i \geq 0$	canonical	= Du Val (A,C,E); A is cyclic quotient $\frac{1}{n}(1,-1)$	$\supseteq$ some toric	$\supseteq$ some toric
$\forall a_i >$ -1	log-terminal	=quotient Iliev, 86 (A,C,E);	$\supseteq$ *quotient* $\supseteq$ toric $\mathbb{Q}$-Gorenstein	$\supseteq$ *quotient* $\supseteq$ toric $\mathbb{Q}$-Gorenstein
$\forall a_i \geq$ -1	log-canonical	classified, Alexeev,92		

rational; Kawamata-Vieweg Vanishing Theorem applies (1982).

Proof of the existence of terminal flips in dim 3 by Mori (1988) relied on the classification of 3-dimensional terminal singularities. For this and other work on MMP he was awarded the Fields Medal in 1990.

Toric Varieties, Toric Singularities

1) Affine Toric Variety

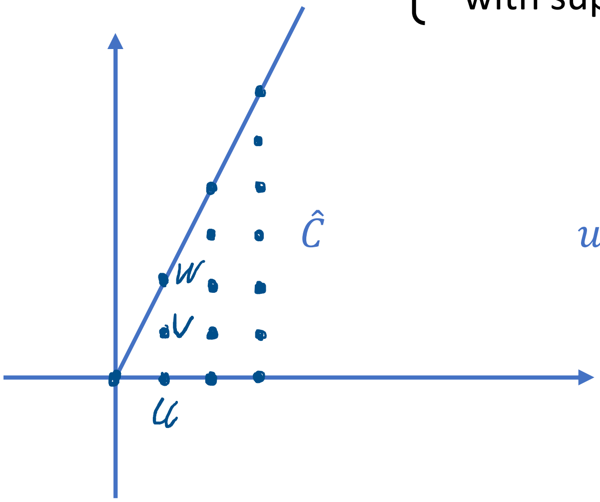
M = ”Lattice of monomials” $\cong$ $\mathbb{Z}^d$

Ex. d=2 $\begin{pmatrix} 3 \\ -2 \end{pmatrix} \leftrightarrow x_1^3\, x_2^{-2}$

Given a closed rational polyhedral cone $\hat{C} \subseteq M \otimes \mathbb{R}$,

affine toric variety $X_{\hat{C}}$= $Spec$ ($\mathbb{C}$ [$\hat{C} \cap M$])

$\left\{ \begin{array}{c} \text{Laurent polynomials} \\ \text{with support in } \hat{C} \end{array} \right\}$



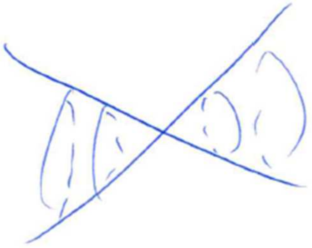
$uw = v^2$

$X_{\hat{C}}$= $Spec$ ($\mathbb{C}$[u, v, w]/(uw - v^2))

ordinary double point

Du Val singularity A_1

“Real” picture: cone



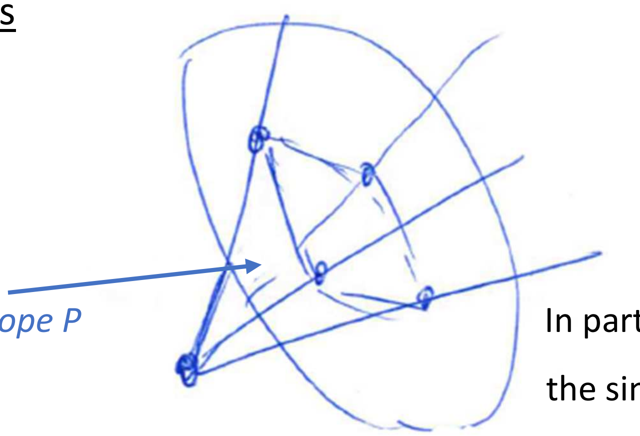
2) General Toric Varieties

To properly glue together affine toric varieties, it is best to look at the dual lattice

N= Hom ($M, \mathbb{Z}$) ($\cong \mathbb{Z}^d$, “lattice of 1-dim. subgroups of $(\mathbb{C}^*)^d$ ”)

Dual Cone C = { $y \in N$ | $\forall\, x \in \hat{C}$ $y(x) \geq 0$ }

Singularities



Convex polytope P

$\mathbb{Q}$ - Gorenstein

same hyperplane

In particular, for a simplicial C

the singularity is $\mathbb{Q}$ - Gorenstein

(in fact, $\mathbb{Q}$ - factorial)

Terminal: *P* is lattice-free

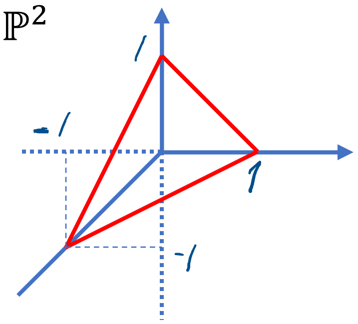
Canonical: points are allowed on the hyperplane only

Log-terminal: all of them

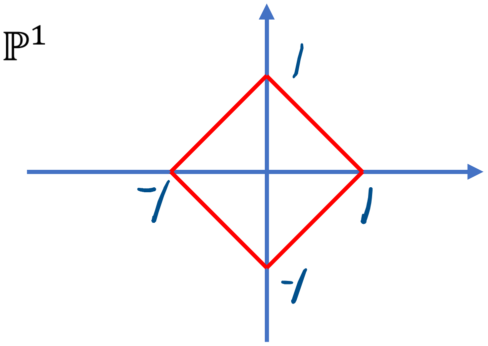
ε -log-terminal (∀ a_i > -1 + ε): ε *P* ∩ N = {0}
(for 0 < ε < 1)

Complete (compact) toric variety ↔ complete fan

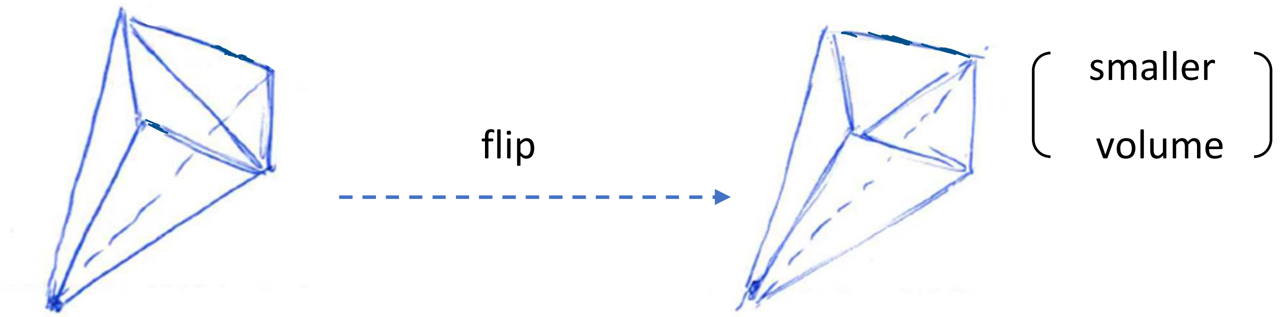
Ex. $\mathbb{P}^2$



$\mathbb{P}^1 \times \mathbb{P}^1$



Toric MMP in dimension 3



divisorial contraction ↔ "removing a vertex"

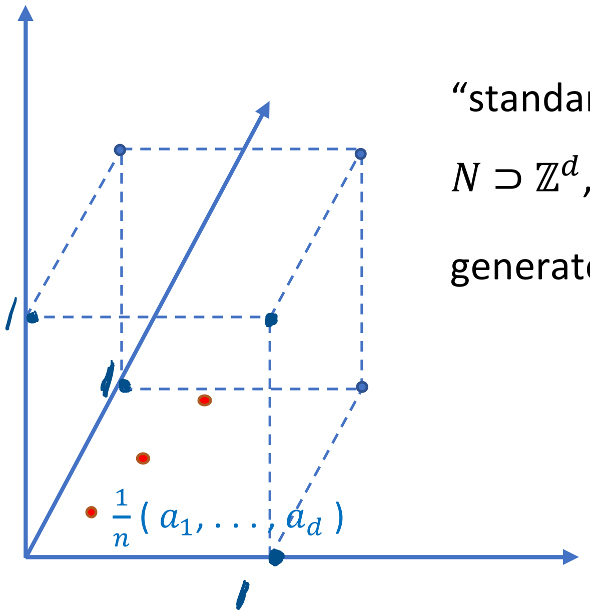
Cyclic Quotient Singularities

$\mu_n = \{ z \in \mathbb{C} \mid z^n = 1 \}$

$\mu_n : \mathbb{C}^d \circlearrowleft$ locally can assume linear action, basis of eigenvectors

$\mu (x_1, \dots , x_d) = (\mu^{a_1} x_1, \dots , \mu^{a_d} x_d) \quad \forall \mu \in \mu_n$

type $\frac{1}{n} (a_1, \dots , a_d)$ all a_i are modulo n



"standard" cone C

$N \supset \mathbb{Z}^d, \quad | N / \mathbb{Z}^d | = n,$

generated by $[\frac{1}{n} (a_1, \dots , a_d)]$

Terminal abelian quotients $\longleftrightarrow$ lattice-free simpices

Classification Results

Dim 3. Lattice-free simplices correspond to type $\frac{1}{n}(1, a, -a)$

Dim 4. Mori, Morrison, Morrison, 1988

Purely experimental paper: cyclic quotients of prime order, for *p*<10,000. Observed and conjectured some patterns, in particular 29 1-parameter "families" of "stable quintuples".

Ex. $\frac{1}{p}$(1,9,-2,-3,-5)

(remove one to get a 4-dim terminal cyclic quotient, for all $p \geq 11$)

Also, a long list of "exceptional quintuples", for *p* < 421

Sankaran, 1989: the list of stable quintuples is complete

A.B, 2000: All but finitely many cyclic quotients of prime order are in MMM list.

Bober, 2009: more conceptual proof of Sankaran's result and more; (Connections to algebraic hypergeometric functions) Dynkin diagrams!

A.B., 2008 Connection to Riemann Hypothesis (Nyman-Beurling-Báez-Duarte Criterion)

"Final Answer"! Valiño-Santos, 2019

The answer is complete; some redundancy (some families may have common members). The approach is based on looking into projections to lower dimensions.

Birational geometry consequences of MMM paper: too many examples. So the theory went a different route: maximalist's approach instead of a minimalist's approach. (We work with the worst possible singularities that we can, barely, handle instead of the smallest class of singularities that we have to allow).

Good news: many new results.

Dimension 3 log-terminal MMP: Shokurov; Kollár et al (1992).

Higher dimensions: Hacon-McKernan-Xu, 2014 (almost everything)

Most recent: Birkar proved BAB Conjecture (2019): boundedness of Fano varieties of dimension *d* with ε-log-terminal singularities. For this and his other MMP work he got Fields Medal in 2018.

Some background on the conjecture. Fano varieties are those where (-K) is ample. The conjecture was proposed around 1993 (Alexeev, A.B., independently). Main evidence: smooth, dimension *d* (Kollár-Miyaoka-Mori, 1992); surfaces (Nikulin, 1989, Alexeev, 1994); toric varieties (A.B.-L.B., 1992)

Toric case: There are only finitely many convex lattice polytopes *P* such that $\varepsilon P \cap \mathbb{Z}^d = \{0\}$. First proved by Hensley, 1983; explicit bounds by Lagarias-Ziegler, 1991. There are certainly many people here who know more about these than I do.

Polytopes with 1 point inside have been studied a lot, in particular by Kasprzyk. These are “easier” than the lattice-free case: a finite list is always easier to get than a finite list of infinite families.

Not that it is actually easy. Great example:

Skarke, Kreuzer, 2000 Classification of 4-dim reflexive (Gorenstein) polytopes. There are 473,800,776 of them.

THANK YOU!