Measuring the Polygons

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Classical Euclidean Geometry

Any triangle is completely determined (up to congruency) by three measurements:

1) three sides
2) side-angle-side
3) angle-side-angle (also angle-angle-side)

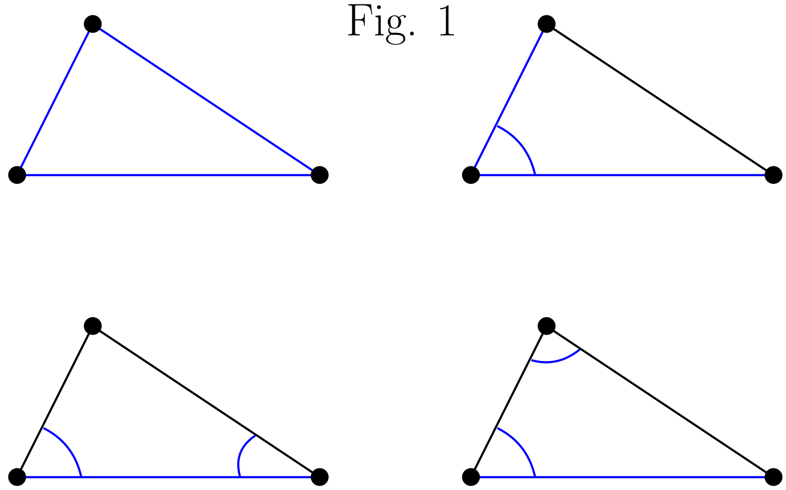


Fig. 1

What about quadrilaterals?

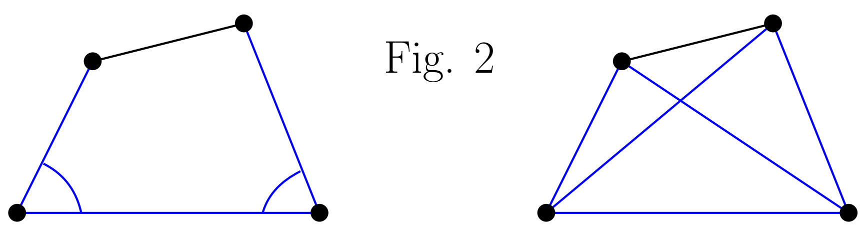


Fig. 2

3 distances, 2 angles   5 distances

One can also do 4 distances and 1 angle, 2 distances and 3 angles, or 1 distance and 4 angles, for the total of 5 measurements.

Question. How many measurements are needed to completely determine n-gons, for $n \geq 4$?

Caution! Need to specify what measurements are allowed. Otherwise, any triangle $\triangle A_1A_1A_3$ with sides a, b, and c is uniquely determined by the "measurement"

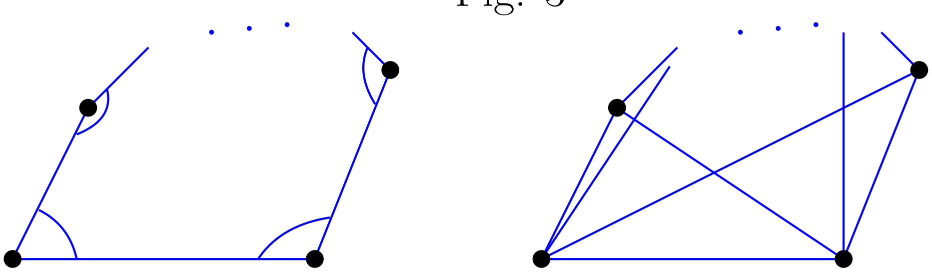
$$(|A_1A_2| - a)^2 + (|A_1A_3| - b)^2 + (|A_2A_3| - c)^2$$

being equal to 0.

Simple Measurements:
1) distance between two vertices: $|A_iA_j|$
2) angle involving three vertices: $\angle A_iA_jA_k$

Answer. One can determine any n-gon by (2n-3) simple measurements. Moreover, (2n-3) distance measurements suffice.

Fig. 3



(n-1) distances, (n-2) angles (2n-3) distances

It is relatively easy to show that for a **generic** n-gon fewer then (2n-3) measurements won't be enough.

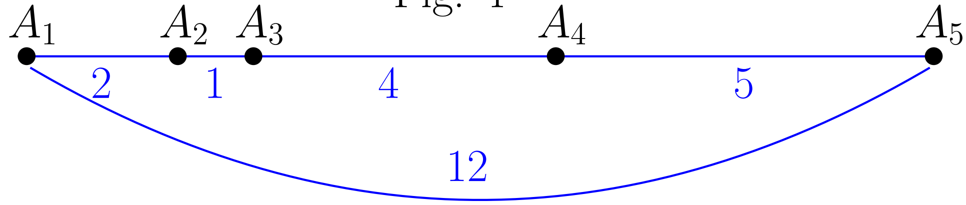
Idea of the proof

Suppose $A_1A_2...A_n$ is a generic n-gon, and suppose it is completely determined by some collection of at most (2n-4) distance and angle measurements. We need at least one distance measurement, otherwise we can scale. We can assume that it is $|A_1A_2|$. We can assume that the points A_1 and A_2 are fixed on the plane, at the correct distance, and the remaining (n-2) points are *arbitrary*. Since each of these points has 2 coordinates, this gives us 2(n-2)=(2n-4) independent variables. But, since we have already used one measurement, we have at most (2n-5) simple measurements to determine these (2n-4) coordinates, and it is impossible.

Question. Can any **specific** n-gon be described by fewer than $(2n-3)$ measurements?

Evidence for a Positive Answer

Fig. 4



$12 = |A_1A_5| \leq |A_1A_2| + |A_2A_3| + |A_3A_4| + |A_4A_5| = 2 + 1 + 4 + 5 = 12$

So the orange inequality is equality, which forces the points to be on one line, and in the correct order!

Definition. We will call the n-gons that can be described by fewer than $(2n-3)$ simple measurements special.

Theorem. All squares are special!

Suppose $|AB| = |AD| = s$, $|AC| = s\sqrt{2}$, and $\angle BCD = 90°$. Then $ABCD$ is a square of side s.

Proof.

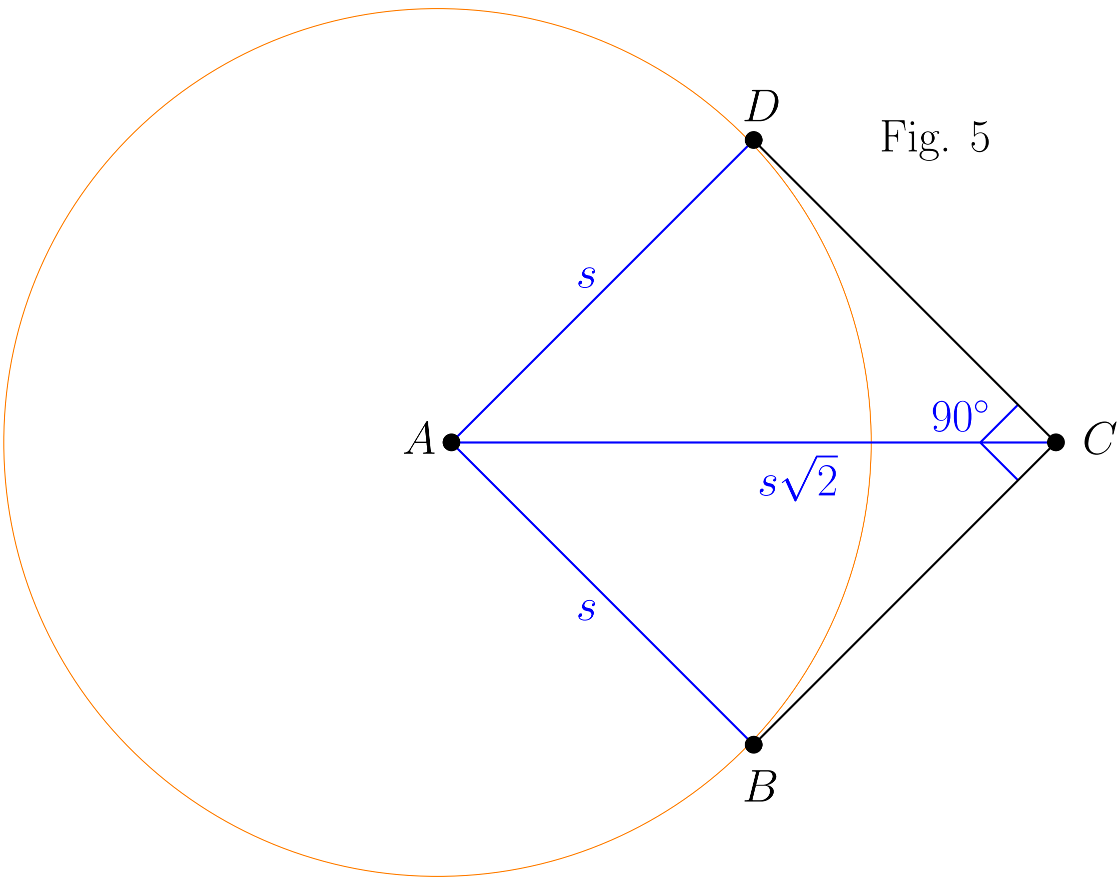


Fig. 5

Fix A and C on the plane. Then B and D lie on the orange circle. The $90°$ angle is the **largest possible angle** between C and any two points on the orange circle. This forces the lines CB and CD to be tangent to orange circle, forcing $ABCD$ to be a square.

This can be generalized to all quadrilaterals $ABCD$ with $\angle ABC = \angle ADC = 90°$.

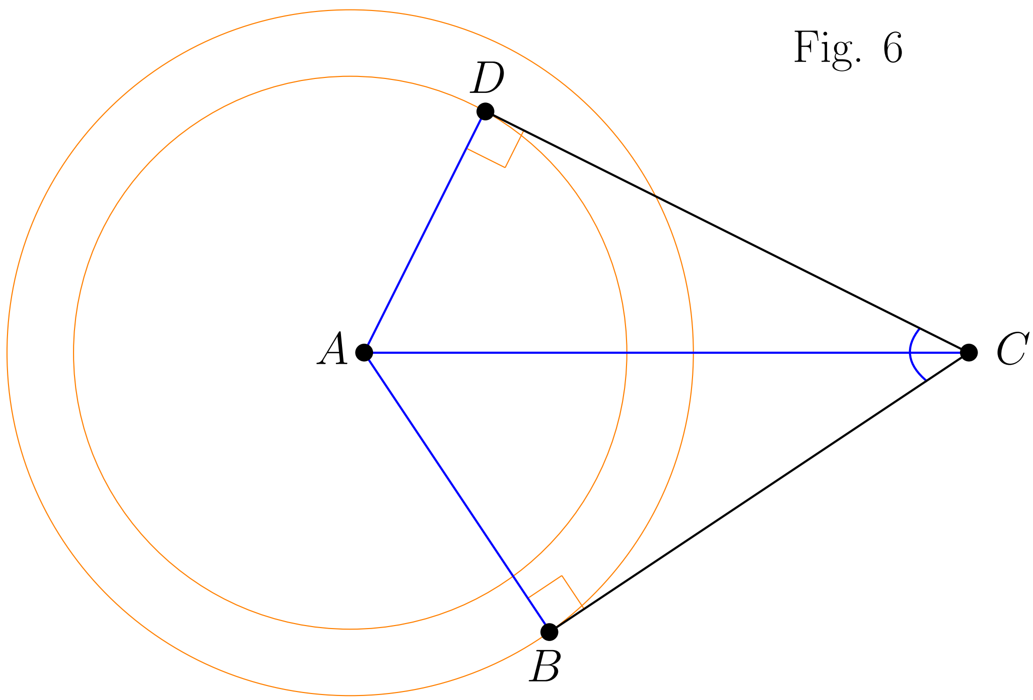


Fig. 6

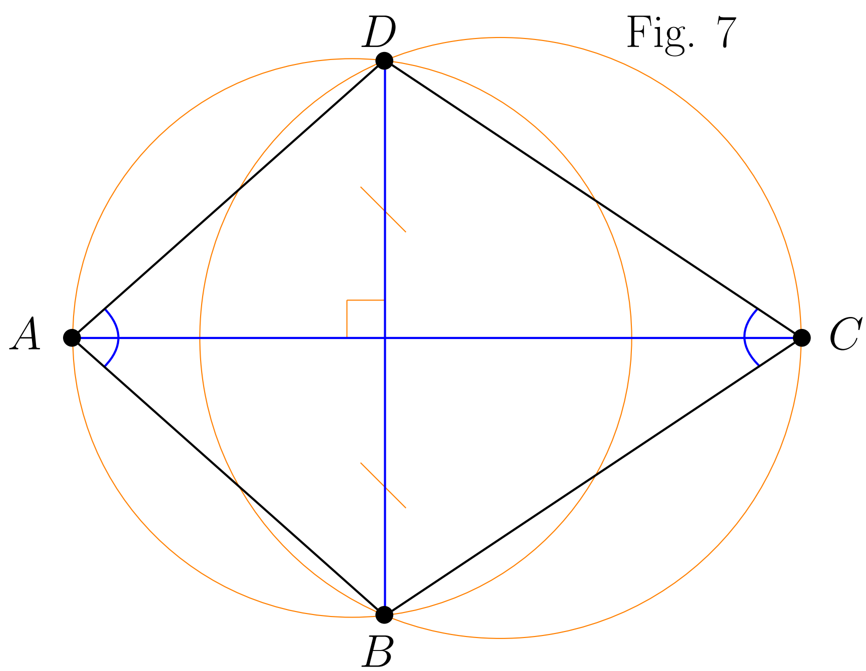
Like in the example for the square, the angle $\angle BCD$ is the largest possible, for any given $|AB|$, $|AC|$, and $|AD|$.

Remark. This family of quadrilaterals includes all rectangles.

Another Construction

All quadrilaterals $ABCD$ with acute angles $\angle BAD$ and $\angle BCD$ with AC being a perpendicular bisector of BD are special.

Fig. 7



If B and D are fixed, then A and C must lie on the corresponding orange circles, by the Inscribed Angle Theorem. And then $|AC|$ is the largest possible when A and C are like in the above picture.

Remark. Such quadrilaterals are known as kites. This family includes all rhombi.

Question. What about n-gons? Specifically, for every given n, what is the smallest number of simple measurements that can completely describe **some** n-gon?

Answer. 1) There exist "superspecial" n-gons that can be described by just n measurements.

2) No n-gon can be described by less than n measurements.

Superspecial Polygons

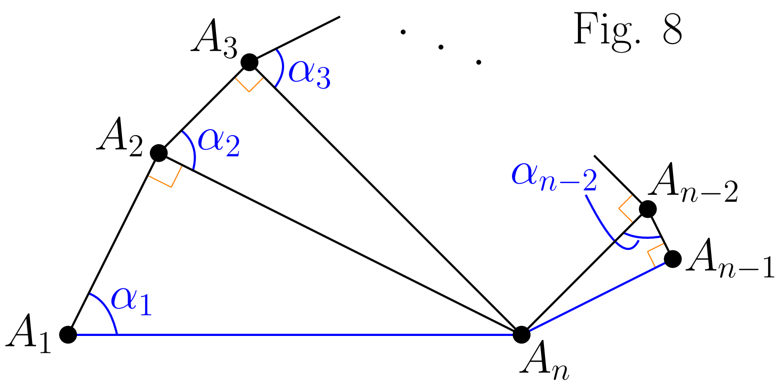


Fig. 8

From the Law of Sines for $\triangle A_n A_1 A_2$,

$|A_n A_2| = |A_n A_1| \cdot \frac{\sin \alpha_1}{\sin \angle A_1 A_2 A_n} \geq |A_n A_1| \cdot \sin \alpha_1$

Likewise,

$|A_n A_3| \geq |A_n A_2| \cdot \sin \alpha_2 \geq |A_n A_1| \cdot \sin \alpha_1 \sin \alpha_2$

$\cdots$

$|A_{n-1} A_n| \geq |A_n A_1| \cdot \sin \alpha_1 \sin \alpha_2 ... \sin \alpha_{n-2}$

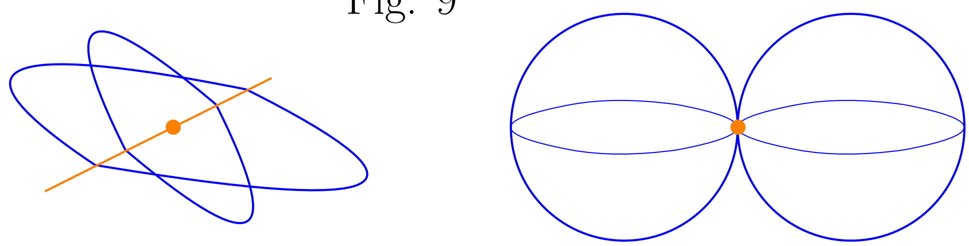
If the above inequality is actually an equality, so are all the intermediate inequalities. So all orange angles are 90°, which determines the polygon!

Lower Bound on the Number of Measurements

Main Idea: Linear Algebra

In dimension 3, two intersecting **planes** must intersect by a line, while two **spheres** (or a sphere and a plane) may intersect by a single point.

Fig. 9



In general, in any dimension m, any $k < m$ **linear** equations that have a common solution must have infinitely many common solutions. Each of the simple measurements gives a non-linear equation on the set of all configurations, but inside of it lies a subset that is determined by two linear equations. The total number of dimensions for the possible configurations is about $2n$, and less than n measurements will leave some "wiggle room".

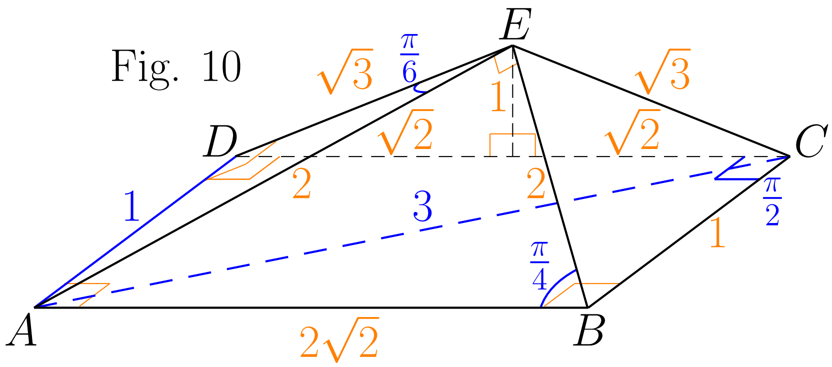
Linear Algebra is the most applicable branch of Mathematics, fully worth learning for any STEM major.

Most problems in modern world involve **many** variables and equations, and the only systems of equations that we can deal with comfortably are systems of **linear** equations.

Dimension 3

"Tent"

Fig. 10



Claim. $ABCDE$ is completely determined by the five blue measurements.

Proof.

1) Since $|AD| = 1$ and $\angle AED = \frac{\pi}{6}$, $|AE| \leq 2$, with the equality if and only if $\angle ADE = \frac{\pi}{2}$.

2) Since $|AE| \leq 2$ and $\angle ABE = \frac{\pi}{4}$, $|AB| \leq 2\sqrt{2}$, with the equality if and only if $|AE| = 2$ and $\angle AEB = \frac{\pi}{2}$.

3) Since $|AD| = 1$ and $|AC| = 3$, $\angle ACD \leq \arcsin\frac{1}{3}$, with the equality if and only if $\angle ADC = \frac{\pi}{2}$.

4) Since $|AB| \leq 2\sqrt{2}$ and $|AC| = 3$, $\angle ACD \leq \arcsin\frac{2\sqrt{2}}{3}$, with the equality if and only if $|AB| = 2\sqrt{2}$ and $\angle ABC = \frac{\pi}{2}$.

5) Since $\angle ACD \leq \arcsin\frac{1}{3}$ and $\angle ACB \leq \arcsin\frac{2\sqrt{2}}{3}$, $\angle BCD \leq \arcsin\frac{1}{3} + \arcsin\frac{2\sqrt{2}}{3} = \frac{\pi}{2}$, with the equality if and only if $\angle ACD = \arcsin\frac{1}{3}$, $\angle ACB = \arcsin\frac{2\sqrt{2}}{3}$, and the ray $[CA)$ lies in the plane of BCD, between the rays $[CB)$ and $[CD)$.

The claim follows from the above considerations.

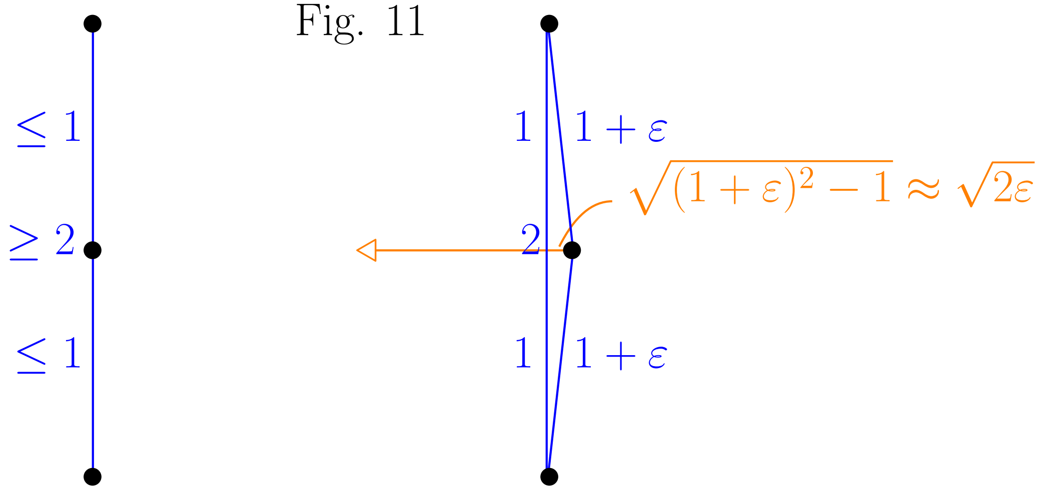
This is more than just a curiosity.

These "extremal" configurations share two remarkable properties:

1) Some of the equalities can be replaced by inequalities;

2) A small change of some parameters of the system results in a relatively large change of some other parameters.

Fig. 11



For example, if $\varepsilon = 0.01$, $\sqrt{2e} \approx 0.14$, which is 14 times larger!

At a very basic level, this is what allows a bow to transfer the large tension force of the bowstring to the high speed of an arrow.

Distance-special Polygons

Our first example of a special configuration of points was points one one line, with only distance measurements. One might think that angle measurements are needed to construct special polygons. But this is only partially true. In fact, the regular octagon can be described by 12 distance measurements, instead of the expected $2 \cdot 8 - 3 = 13$. The picture below shows one such way. Another way can be described as "8 sides plus 4 main diagonals".

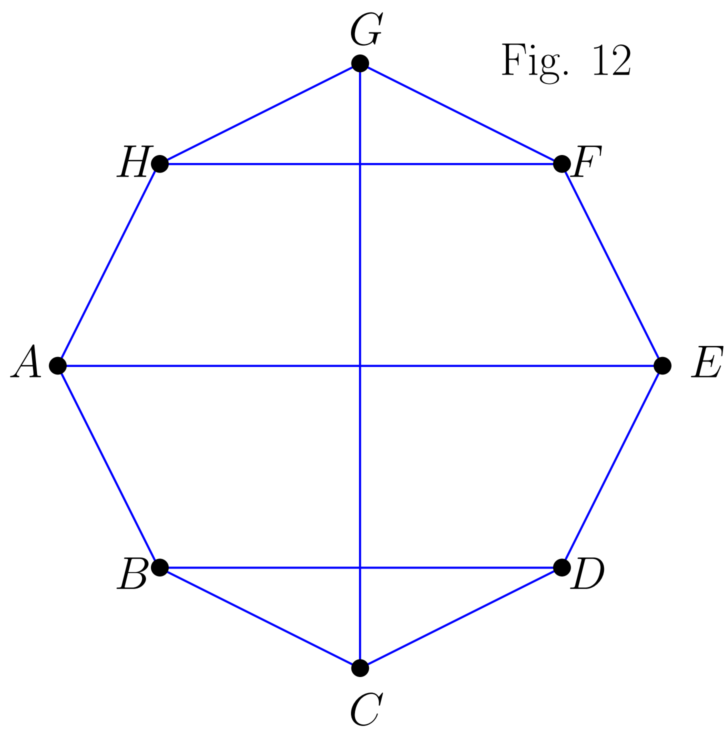


Fig. 12

This is the smallest such example, if we only consider configurations of points on the plane with no three points on the same line. For $n \leq 7$ such configurations cannot be distance-special. For $n \geq 8$, the smallest number of distance-only measurements is $\lceil \frac{3n}{2} \rceil$, that is $\frac{3n}{2}$ for even n and $\frac{3n+1}{2}$ for odd n.

Acknowledgments. This talk is based on joint work with Mark Dickinson and Stuart Hastings, specifically on the last section of our paper:

A. Borisov, M. Dickinson, S. Hastings. A Congruence Problem for Polyhedra. *American Mathematical Monthly*, **3** (2010) 232–249.

If you are interested in learning more, feel free to email me at

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Thank you!