Bi-Euclidean Spaces and Coherent Sheaves on Arakelov Curves

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1) Some History

Bernhard Riemann:

1857: Riemann's Inequality (strengthened by Gustav Roch to become Riemann-Roch Theorem)

1959: Functional Equation for ζ.

Did Riemann and/or Roch see a connection?

1882: Julius Wilhelm Richard Dedekind,
Heinrich Martin Weber

1920s: Emil Artin, Erich Hecke,...

1931: Friedrich Karl Schmidt

1950: John Tate, Kenkichi Iwasawa

2) 2000: Gerard van der Geer, René Schoof

F is a number field, D is an Arakelov divisor on $\overline{Spec(O_F)}$

$D \rightsquigarrow (I_D, |\cdot|_D)$ (fractional ideal, quadratic function)

$h^0(D) = \log(\sum_{x \in I_D} e^{-\pi|x|_D})$

$h^0(D) - h^0(K-D) = \deg D - \frac{1}{2}\log|\Delta_F|$

$h^1(D) := h^0(K-D)$

2003: A.B. (adelic version: Ichiro Miyada)

$H^0(D)$ generalization of loc. comp. abelian group, elements "partially exist" using convolution of measures structures. $H^1(D)$ elements "not precisely positioned"; determined by Čech cohomology-like construction.

$H^1(D) = \widehat{H^0(K-D)}$ (Pontryagin duality)

$h^0(D) - h^1(D) = \deg D - \frac{1}{2}\log|\Delta_F|$

The following picture represents the ghost-space $\mathbb{R}_{e^{-\pi x^2}}$. One should think of it as being embedded into the usual real line.

0

The quotient $\mathbb{R}/\mathbb{R}_{e^{-\pi x^2}}$ is the space $\mathbb{R}^{\mu}$, where μ is the probability measure $e^{-\pi x^2} dx$.

The short exact sequence of ghost spaces

$$0 \longrightarrow \mathbb{R}_{e^{-\pi x^2}} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^{e^{-\pi x^2} dx} \longrightarrow 0$$

is the $\mathbb{R}$ analog of the short exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

In particular, it is Pontryagin self-dual.

3) 2017 Thomas McMurray Price;

Oded Regev, Noah Stephens-Davidowitz

M, N are sublattices in Z^n, with a positive quadratic function Q. Define $\dim(M) = \log(\sum_{x \in M} e^{-\pi Q(x)})$. Then

$$\dim(M+N) + \dim(M \cap N) \geq \dim(M) + \dim(N)$$

Proof is short, but not obvious. It is **false** for general positive positive-definite functions instead of $e^{-\pi Q(x)}$.

Euclidean lattices = " loc. free sheaves on $\overline{Spec(\mathbb{Z})}$"

Morphisms: maps of lattices s.t. the corresponding linear maps (tensored by $\mathbb{R}$) are **non-expanding**.

4) A.B. - Jaiung Jun, work in progress

For now, all spaces are finite-dimensional...

Def. A generalized real Euclidean space is a vector space over $\mathbb{R}$ with a non-negative quadratic function.

Ex. $(\mathbb{R}, x^2) \subseteq (\mathbb{R}, 0)$. Here $(\mathbb{R}, 0) = \mathbb{R}$; we denote $(\mathbb{R}, x^2)$ by $O_{\mathbb{R}}$. We call it a **hoop** (Russian: **обруч**).

Euclidean spaces are free modules over $O_{\mathbb{R}}$.

Notation. $V = (\underline{V}, |\cdot|_V)$, where $|x|_V = Q(x)$ is quadratic, and $\underline{V}$ is the **underlining vector space**, viewed as $\underline{V} = V \otimes_{O_{\mathbb{R}}} \mathbb{R}$.

Two ways to visualize them.

Particle: $V = (\underline{V}, |\cdot|_V) \leftrightarrow B_V = \{x \in \underline{V} \mid |x|_V \leq 1\}$

Wave: V is $\underline{V}$, but with elements existing with probablility $e^{-\pi Q(x)}$.

Def. A morphism $f : V_1 \to V_2$ is an $\mathbb{R}$-linear map $\underline{f} : \underline{V_1} \to \underline{V_2}$ s.t. $\forall x \in \underline{V_1} \quad |\underline{f}(x)|_{V_2} \leq |x|_{V_1}$.

This category contains vector spaces as a full subcategory: $\underline{V} = (\underline{V}, 0)$.

Def. $Im(f) = (Im(\underline{f}), Q)$, where $Q(y) = \inf\limits_{\underline{f}(x)=y} |x|_{V_1}$

Def. $Ker(f) = (Ker(\underline{f}), |\cdot|_{V_1})$ (its restriction)

$$0 \to Ker(f) \to V_1 \to Im(f) \to 0$$

Moreover, the sequence splits: $V_1 \cong Ker(f) \oplus U$, where U is the orthogonal complement of $Ker(\underline{f})$ in V, which is isomorphic fo $Im(f)$.

5) Main challenge: quotients.

$V_1 \subseteq V_2$. What is V_2/V_1? These will be fibers at infinity of coherent sheaves on $\overline{Spec(\mathbb{Z})}$. Quotient of an ellipsoid by a smaller ellipsoid. Insanity? Yes and no.

Def. $W \subset V$ (strict inclusion) means that $\forall x \neq 0$ $|x|_W > |x|_V$. Much stronger than $W \subsetneq V$. The $\mathbb{Z}_p$ analog of this is $W \subseteq pV$.

Def. Bi-Euclidean space is a pair of Euclidean spaces (V, W) with $W \subset V$. We think of it as V/W.

Generalization: Suppose $W \subseteq V$. Then the set $\{x \in \underline{W} \mid |x|_W = |x|_V\}$ is an $\mathbb{R}$-vector space, and its orthogonal complement in W is contained in its orthogonal complement in V. We define V/W as the quotient of these orthogonal complements:

$$V/W = (V' \oplus U)/(W' \oplus U) = V'/W' = (V', W')$$

Def. A morphism $\bar{f} : (V_1, W_1) \to (V_2, W_2)$ is such $f : V_1 \to V_2$ that its restriction to W_1 is a map from W_1 to W_2. An isomorphism is an invertible morphism: both f and its restriction to W_1 are isometries.

Def. $(V_1, W_1) \oplus (V_2, W_2) = (V_1 \oplus V_2, W_1 \oplus W_2)$

Ex. $V = (V, \{0\})$, in particular $O_{\mathbb{R}} = O_{\mathbb{R}}/\{0\}$

Ex. $(O_{\mathbb{R}}, aO_{\mathbb{R}})$, for $0 < a < 1$, is a "cyclic $O_{\mathbb{R}}$-module of finite length"; $length(O_{\mathbb{R}}/aO_{\mathbb{R}}) = -\log a$.

Thm. Every (V, W) is isomorphic to $\bigoplus_{i=1}^{n}(O_{\mathbb{R}}/a_iO_{\mathbb{R}})$, with the unique multiset $\{a_i\}$, $a_i \geq 0$.

Appeared in 2018 PhD thesis of Patrick Milano as "Gaussian mixed ghost-spaces"

Considering separately the $a_i = 0$, it is a direct sum of the free part and the torsion.

$$V/W \cong O_{\mathbb{R}}^k \oplus \bigoplus_{i=1}^{m} (O_{\mathbb{R}}/a_i O_{\mathbb{R}})$$

Moreover, the torsion is $(\underline{W}, |\cdot|_V)/W$, and the free part is the orthogonal complement of $\underline{W}$ in V.

Every morphism of bi-Euclidean spaces induces a morphism of their torsion submodules.

So far, so good. But we want $Im(\bar{f})$, $Ker(\bar{f})$, tensor products, etc. For this we need to go back to Euclidean spaces and define their intersection and sum.

Def. Suppose U and V are Euclidean subspaces of $\underline{W}$. We define $U \cap V$ and $U + V$ as follows. Consider a "common orthogonal basis" of U and V: $\{x_1, ..., x_n\}$ s.t.

1) $\{x_1, ..., x_n\}$ is a basis of $\underline{U} + \underline{V}$;

2) some subcollection of $\{x_1, ..., x_n\}$ forms an orthogonal basis of U;

3) some subcollection of $\{x_1, ..., x_n\}$ forms an orthogonal basis of V.

Suppose $|x_i|_U = a_i$, $|x_i|_V = b_i$, $+\infty$ when undefined. Then $U \cap V$ is given by orthogonal basis of x_i with $|x_i|_{U\cap V} = max(|x_i|_U, |x_i|_V)$ and $U + V$ is given by orthogonal basis of x_i with $|x_i|_{U+V} = min(|x_i|_U, |x_i|_V)$.

Alternatively, $|x|_{U+V} = inf(|u|_U + |v|_V)$ over pairs $(u, v) \in (\underline{U}, \underline{V})$ s.t. $u+v = x$ and orthogonal projections of u and v to $\underline{U} \cap \underline{V}$ are both U- and V-orthogonal. And then $|x|_{U \cap V} = |x|_U + |x|_V - |x|_{U+V}$.

Geometrically: for Euclidean spaces $B_{U \cap V}$ is the elipsoid of the largest volume in $B_U \cap B_V$; B_{U+V} is the ellipsoid of the smallest volume that contains B_U and B_V.

Good News: 1) $U \cap V \subseteq U \subseteq U + V$

2) $U \subseteq V \Leftrightarrow U \cap V = U \Leftrightarrow U + V = V$

Bad News:

$$1) \begin{cases} W \subseteq U \\ W \subseteq V \end{cases} \not\Rightarrow W \subseteq U \cap V; \quad \begin{cases} W \supseteq U \\ W \supseteq V \end{cases} \not\Rightarrow W \supseteq U + V$$

2) The operations are **not associative**.

Many things still work, but the intuition needs to *stretch*.

Lemma. Suppose $W, V' \subseteq V$. Then $V'/(V' \cap W) = (V' + W)/W$. Note that if $W \subseteq V' \subseteq V$, this is V'/W.

Even though in general $V' + W \not\subseteq V$, we consider the above space a subspace of V/W. Specifically:

Def. Suppose $W \subset V$. A bi-Euclidean space (V', W') is a subspace of (V, W) if there exists a (usual) subspace $\underline{U}$ of $\underline{V}$ such that $W' = W \cap \underline{U}$ and $W' \subset V' \subseteq V \cap \underline{U}$. In this case, we write $(V', W') \subseteq (V, W)$.

This notion is transitive: if $(V', W') \subseteq (V, W)$ and $(V'', W'') \subseteq (V', W')$, then $(V'', W'') \subseteq (V, W)$. There are also natural inequalities for the a_i in the cyclic decomposition of bi-Euclidean spaces and their subspaces.

Image and Kernel

$$\begin{array}{ccccccccc} 0 & \to & W_1 & \to & V_1 & \to & V_1/W_1 & \to & 0 \\ & & \downarrow & & \downarrow f & & \downarrow \bar{f} & & \\ 0 & \to & W_2 & \to & V_2 & \to & V_2/W_2 & \to & 0 \end{array}$$

$Im(\bar{f}) = f(V_1)/(f(V_1) \cap W_2)$, where $f(V_1) = Im(f)$
$Ker(\bar{f}) = f^{-1}(W_2)/(f^{-1}(W_2) \cap W_1)$, where
$f^{-1}(W_2) = (\underline{f})^{-1}(W_2) \cap V_1$, $|x|_{(\underline{f})^{-1}(W_2)} = |\underline{f}(x)|_{W_2}$

Good News: 1) Image and Kernel are subspaces;

2) For the Euclidean spaces, we recover old definitions;

3) Natural notions of injective and surjective.

Bad News: 1) There is no decomposition of a map into surjective, followed by injective: need some equivalence relation on maps.

2) Not all subgroups have natural quotients, even if they are kernels of maps.

3) In general, no First Isomorphism Theorem.

Need more sophisticated notions of morphisms. One idea: objects are equivalence classes of pairs (V, W) with $W \subseteq V$. Morphisms are equivalence classes of morphisms of resolutions...

6) We do have tensor products and more.

Def. $V_1 \otimes_{O_{\mathbb{R}}} V_2 = (\underline{V_1} \otimes \underline{V_2}, Q)$, where

$$Q(v_1 \otimes v_2) = |v_1|_{V_1} \cdot |v_2|_{V_2}$$

$$V_1/W_1 \otimes_{O_{\mathbb{R}}} V_2/W_2 = V_1 \otimes_{O_{\mathbb{R}}} V_2/(V_1 \otimes_{O_{\mathbb{R}}} W_2 + W_1 \otimes_{O_{\mathbb{R}}} V_2)$$

Theorem. $(\bigoplus_{i=1}^{n} O_{\mathbb{R}}/a_iO_{\mathbb{R}}) \otimes_{O_{\mathbb{R}}} (\bigoplus_{j=1}^{m} O_{\mathbb{R}}/b_jO_{\mathbb{R}})$ is naturally isomorphic to $\bigoplus_{i=1,j=1}^{n,m} O_{\mathbb{R}}/max(a_i,b_j)O_{\mathbb{R}}$

Crazier idea. For modules over a ring, Hom is itself a module over that ring, not just a set. Something like that works for the hoops too.

Def. For a Euclidean space V, the dual module $V^*{=}(Hom_{O_{\mathbb{R}}}(\underline{V},\mathbb{R}),||f||^2)$; $Hom_{O_{\mathbb{R}}}(V_1,V_2){=}V_1^*\otimes_{O_{\mathbb{R}}}V_2$

Interestingly, $B_{Hom_{O_{\mathbb{R}}}(V_1,V_2)} \subseteq Hom(V_1,V_2)$, because the usual norm of a matrix is less than or equal to its Hilbert-Schmidt norm. But in general they are not equal.

One can also define $Hom_{O_{\mathbb{R}}}$ for bi-Euclidean spaces as a bi-Euclidean space.

7) What's next?

1) Coherent sheaves: lattices in bi-Euclidean spaces. Likely, all needed inequalities to define h^0 and h^1 are already in RS-D paper.

2) Extend Arakelov geometry to bi-Hermitian sheaves at infinity. Note: in classical Arakelov geometry for a horizontal curve P and an Arakelov divisor D no map of invertible Hermitian sheaves $O_X(D-P)\to O_X(D)$, ultimately because the Green function is normalized to integrate to 0 (which also causes some negative intersections of horizontal divisors). IMHO, this is just an unfortunate convention, that is easy to fix.

3) For $f: X\to\overline{Spec(\mathbb{Z})}$, are f_*, R^if_* coherent sheaves?

References

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THANK YOU!